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Unique common fixed point theorems for single and set valued D-maps

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ABSTRACT

In this paper, we obtain two unique common fixed point theorems for two pairs of sub compatible D-maps satisfying weak contractive condition in a metric space. Our results generalize and extend the theorems of [2] and [10] to the setting of two pairs of single and set-valued maps.

Introduction and Preliminaries

In this paper (X, d) denotes a metric space and $B(X)$ is the set of all non empty bounded subsets of X. For all A, B in $B(X)$ we define,

$$\delta(A,B) = sup\{d(a,b) : a \in A, b \in B\}.$$

If $A = \{a\}$, $\delta(A,B) = \delta(a,B)$. Also if $B = \{b\}$, $\delta(A,B) = d(a,b)$.

Now we give properties of δ:

$$\delta(A,B) = \delta(B,A) \geq 0.$$
$$\delta(A,B) = \delta(A,C) + \delta(C,B).$$
$$\delta(A,A) = diamA.$$
$$\delta(A,B) = 0 \Leftrightarrow A = B = \{a\}.$$

For all $A, B, C \in B(X)$.

First , we give some known preliminaries.

Definition 1.1. ([5]) A sequence $\{A_n\}$ of nonemty subsets of X is said to be convergent to a subset A of X if

(i) each point a in A is the limit of a convergent sequence $\{a_n\}$ where a_n is in A_n for $n \in N$,

(ii) for arbitrary $\varepsilon > 0$, there exists an integer m such that $A_n \subseteq A_\varepsilon$ for $n > m$, where A_ε denotes the set of all points x in X for which there exists a point a in A, depending on x, such that $d(x,a) < \varepsilon$, A is then said to be the limit of the sequence $\{A_n\}$.

Lemma 1.2. ([5]) If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ converging to A and B in $B(X)$, respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A,B)$.

Lemma 1.3. ([6]) Let $\{A_n\}$ be a sequence in $B(X)$ and y be a point in X such that $\delta(A_n, y) \to 0$. Then the sequence $\{A_n\}$ converges to the set $\{y\}$ in $B(X)$.

In [9] , Sessa et.al. introduced the concept of weak commutativity for single and multivalued maps as follows :

Definition 1.4. ([9]) The maps $F : X \to B(X)$ and $f : X \to X$ are said to be weakly commuting if $fFx \in B(X)$ and $\delta(Ffx, fFx) \leq \max\{\delta(fx, Fx), diam\, fFx\}$, for all $x \in X$.

Further, Liu Li Shan [8], extended the above definition as follows :

Definition 1.5. ([8]) The maps $F : X \to B(X)$ and $f : X \to X$ are said to be δ- compatible if $\lim_{n\to\infty} \delta(Ffx_n, fFx_n) = 0,$ whenever $\{x_n\}$ is a sequence in X such that $fFx_n \in B(X)$, $Fx_n \to \{t\}$ and $fx_n \to t$ for some t in X.

Afterwards, Jungck and Rhoades [7], gave a generalization of the above definition as follows:

Definition 1.6. ([7]) The maps $F : X \to B(X)$ and $f : X \to X$ are said to be subcompatible if

$$\{t \in X \,/\, Ft = \{ft\}\} \subseteq \{t \in X \,/\, Fft = fFt\}.$$

Obviously two δ-compatible maps are subcompatible but converse is not true (see examples in [3]).

Definition 1.7. ([4]) The maps $F : X \to B(X)$ and $f : X \to X$ are said to be D-maps if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = t$ and $\lim_{n\to\infty} Fx_n = \{t\}$ for some $t \in X$.

Definition 1.8. ([10]) A self map $T : X \to X$ is said to be weakly contractive with respect to a self map $f : X \to X$ if $d(Tx,Ty) \leq d(fx, fy) - \varphi(d(fx, fy))$ for all $x, y \in X$, where $\varphi : [0,\infty) \to [0,\infty)$ is continuous, $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$.

Definition 1.9. ([2]) A self map $T : X \to X$ is said to be a generalized weakly contractive with respect to a self map $f : X \to X$ if

$$d(Tx,Ty) \le \max\left\{ \begin{array}{c} d(fx,fy), d(fx,Tx), d(fy,Ty), \\ \frac{1}{2}[d(fx,Ty)+d(fy,Tx)] \end{array} \right\} - \varphi\left(\max\left\{ \begin{array}{c} d(fx,fy), d(fx,Tx), d(fy,Ty), \\ \frac{1}{2}[d(fx,Ty)+d(fy,Tx)] \end{array} \right\} \right)$$

for all $x, y \in X$, where $\varphi : [0,\infty) \to [0,\infty)$ is continuous, $\varphi(0) = 0$ *and* $\varphi(t) > 0$ for $t > 0$.

Definition 1.10. ([1]) The pair (f, T) is said to satisfy property (E.A) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = t$ *and* $\lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Recently G.V.R.Babu et al. [2, 10], proved the following theorems of common fixed points of a pair of selfmaps.

Theorem 1.11. (Theorem 3.1 of [10]) Let (X, d) be a metric space and let T, $f : X \to X$ be weakly compatible selfmaps satisfying property (E.A). Assume that T is weakly contractive with repect to f. If $f(X)$ is closed, then f and T have a unique fixed point in X.

Theorem 1.12. (Theorem 4.1 of [2]) Let (X,d) be a metric space and let T, $f : X \to X$ be selfmaps satisfying property (E.A). Assume that T is a generalized weakly contractive map with repect to f. If $f(X)$ is closed, then f and T have coincidence points and f and T have a unique point of coincidence in X.

Generally to prove common fixed point theorems for two pairs of maps or Jungck type maps using property (E.A.), one can tempt to assume the closedness of one of the mappings or surjectiveness of one of the mappings. Some times, the authors assume the surjectiveness of two mappings when they used the common property.

In this paper, we relax some conditions by introducing the following two definitions.

Definition 1.13. Let f be a self map on a metric space (X, d) and let $S : X \to B(X)$ be a set-valued map. The pair (f, S) is said to be a pair of D-maps with respect to f, if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = z$ *and* $\lim_{n\to\infty} Sx_n = \{z\}$ for some $z \in f(X)$.

Definition 1.14. Let f, g be self maps on a metric space (X, d) and let $S : X \to B(X)$ be a set-valued map. The pair (f, S) is said to be a pair of D-maps with respect to (f, g), if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = z$ *and* $\lim_{n\to\infty} Sx_n = \{z\}$ for some $z \in f(X) \cap g(X)$.

The aim of this paper is to improve and extend Theorems 1.11 and 1.12 for two pairs of single and set valued maps by using above definitions.

Main Results

Theorem 2.1: Let f, g be self maps on a metric space (X, d) and let S, $T : X \to B(X)$ be set-valued maps such that

$$(2.1.1) \quad \delta(Sx,Ty) \le \max\left\{ \begin{array}{c} d(fx,gy)+\delta(fx,Sx)+\delta(gy,Ty), \delta(fx,Ty), \\ \delta(gy,Sx) \end{array} \right\} - \varphi\left(\max\left\{ \begin{array}{c} d(fx,gy)+\delta(fx,Sx)+\delta(gy,Ty), \\ \delta(fx,Ty), \delta(gy,Sx) \end{array} \right\} \right)$$

for all $x, y \in X$, where $\varphi : [0,\infty) \to [0,\infty)$ is continuous, $\varphi(t) > 0$ for $t > 0$,

(2.1.2) (f, S) and (g, T) are subcompatible pairs.

(2.1.3) (a) (f, S) is a pair of D-maps with respect to f and $Sx \subseteq g(X)$, $\forall\, x \in X$.

(or)

(2.1.3) (b) (g, T) is a pair of D-maps with respect to g and $Tx \subseteq f(X)$, $\forall\, x \in X$.

Then f, g, S and T have a unique common fixed point in X.

Proof. Suppose (2.1.3) (a) holds.

Since (f, S) is a pair of D-maps with respect to f, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = z \textit{ and } \lim_{n\to\infty} Sx_n = \{z\} \quad \text{for some } z \in f(X).$$

Hence there exists $u \in X$ such that $z = fu$.

Since $Sx \subseteq g(X)$ for all $x \in X$ there exist $\alpha_n \in Sx_n$ and $y_n \in X$ such that $\alpha_n = gy_n$ for all n.

Also $d(gy_n, z) = d(\alpha_n, z) \le \delta((Sx_n, z) \to 0$ as $n \to \infty$.

Thus $\lim_{n\to\infty} gy_n = z$.

Now,

$$\delta(Sx_n, Ty_n) \le \max\left\{\begin{matrix} d(fx_n, gy_n) + \delta(fx_n, Sx_n) + \delta(gy_n, Ty_n), \\ \delta(fx_n, Ty_n), \delta(gy_n, Sx_n) \end{matrix}\right\}$$
$$-\varphi\left(\max\left\{\begin{matrix} d(fx_n, gy_n) + \delta(fx_n, Sx_n) + \delta(gy_n, Ty_n), \\ \delta(fx_n, Ty_n), \delta(gy_n, Sx_n) \end{matrix}\right\}\right)$$

Letting $n \to \infty$ we have,

$$\delta(\{z\}, \lim_{n\to\infty} Ty_n) \le \max\left\{d(z,z) + \delta(z,\{z\}) + \delta(\{z\}, \lim_{n\to\infty} Ty_n),\ \delta(\{z\}, \lim_{n\to\infty} Ty_n),\ \delta(z,\{z\})\right\}$$
$$-\varphi\left(\max\ \left\{d(z,z) + \delta(z,\{z\}) + \delta(\{z\}, \lim_{n\to\infty} Ty_n),\ \delta(\{z\}, \lim_{n\to\infty} Ty_n),\ \delta(z,\{z\})\right\}\right).$$

Thus $$\delta(\{z\}, \lim_{n\to\infty} Ty_n) \le \delta(\{z\}, \lim_{n\to\infty} Ty_n) - \varphi(\delta(\{z\}, \lim_{n\to\infty} Ty_n)).$$

Since $\varphi(t) > 0$ for all $t > 0$, we have $\delta(\{z\}, \lim_{n\to\infty} Ty_n) = 0$.

Hence $\lim_{n\to\infty} Ty_n = \{z\}$. Thus

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \{z\},\ \lim_{n\to\infty} fx_n = z = \lim_{n\to\infty} gy_n.$$

Now,

$$\delta(Su, Ty_n) \le \max\left\{\begin{matrix} d(fu, gy_n) + \delta(fu, Su) + \delta(gy_n, Ty_n), \delta(fu, Ty_n), \\ \delta(gy_n, Sx_n) \end{matrix}\right\}$$
$$-\varphi\left(\max\left\{\begin{matrix} d(fu, gy_n) + \delta(fu, Su) + \delta(gy_n, Ty_n), \\ \delta(fu, Ty_n), \delta(gy_n, Sx_n) \end{matrix}\right\}\right).$$

Letting $n \to \infty$ we have,

$$\delta(Su, \{z\}) \le \max\left\{d(z,z) + \delta(z, Su) + \delta(z,\{z\}), \delta(z,\{z\}), \delta(z,\{z\})\right\}$$
$$-\varphi\left(\max\left\{d(z,z) + \delta(z, Su) + \delta(z,\{z\}, \delta(z,\{z\}), \delta(z,\{z\})\ \right\}\right)$$

which implies that

$$\delta(Su,\{z\}) \le \delta(Su,\{z\}) - \varphi(\delta(Su,\{z\})).$$

It follows that $\delta(Su,\{z\}) = 0$. Hence $Su = \{z\}$. So $Su = \{z\} = \{fu\}$.

Since $\{z\} = Su \subseteq g(X)$, there exists $w \in X$ such that $z = gw$.

Now,

$$\delta(Sx_n,Tw)\le \max\left\{\begin{matrix} d(fx_n,gw)+\delta(fx_n,Sx_n)+\delta(gw,Tw),\delta(fx_n,Tw), \\ \delta(gw,Sx_n)\end{matrix}\right\}$$
$$-\varphi\left(\max\left\{\begin{matrix} d(fx_n,gw)+\delta(fx_n,Sx_n)+\delta(gw,Tw), \\ \delta(fx_n,Tw),\delta(gw,Sx_n)\end{matrix}\right\}\right)$$

Letting $n\to\infty$ we have,

$$\delta(\{z\},Tw)\le \max\{d(z,z)+\delta(z,\{z\})+\delta(z,Tw),\delta(z,Tw),\delta(z,\{z\})\}$$
$$-\varphi\left(\max\{d(z,z)+\delta(z,\{z\})+\delta(z,Tw),\delta(z,Tw),\delta(z,\{z\})\}\right)$$
$$=\delta(\{z\},Tw)-\varphi(\delta(\{z\},Tw))$$

which implies that $Tw=\{z\}$. Thus $Su=Tw=\{z\},\ \ fu=gw=z$.

Since (f, S) is subcompatible we have, $Sz=Sfu=fSu=\{fz\}$.

Now,

$$\delta(Sz,Tw)\le \max\{d(fz,gw)+\delta(fz,Sz)+\delta(gw,Tw),\delta(fz,Tw),\delta(gw,Sz)\}$$
$$-\varphi\left(\max\{d(fz,gw)+\delta(fz,Sz)+\delta(gw,Tw),\delta(fz,Tw),\delta(gw,Sz)\}\right)$$

which implies that,

$$\delta(Sz,\{z\})\le \max\{d(fz,z)+\delta(fz,\{fz\})+\delta(z,\{z\}),\delta(Sz,\{z\}),\delta(Sz,\{z\})\}$$
$$-\varphi\left(\max\{d(fz,z)+\delta(fz,\{fz\})+\delta(z,\{z\}),\delta(Sz,\{z\}),\delta(Sz,\{z\})\}\right)$$
$$=\delta(Sz,\{z\})-\varphi(\delta(Sz,\{z\})).$$

It follows that $Sz=\{z\}$.

Thus

$$Sz=\{z\}=\{fz\} \quad\text{..} (\text{ I })$$

Since (g, T) is sub compatible we have $Tz=\{gz\}$.

Now,

$$\delta(Su,Tz)\le \max\{d(fu,gz)+\delta(fu,Su)+\delta(gz,Tz),\delta(fu,Tz),\delta(gz,Su)\}$$
$$-\varphi\left(\max\{d(fu,gz)+\delta(fu,Su)+\delta(gz,Tz),\delta(fu,Tz),\delta(gz,Su)\}\right)$$

which implies that,

$$\delta(\{z\},Tz)\le \max\{d(z,gz)+\delta(z,\{z\})+\delta(Tz,Tz),\delta(z,\{z\}),\delta(Tz,\{z\})\}$$
$$-\varphi\left(\max\{d(z,gz)+\delta(z,\{z\})+\delta(Tz,Tz),\delta(z,\{z\}),\delta(Tz,\{z\})\}\right)$$
$$=\delta(\{z\},Tz)-\varphi(\delta(\{z\},Tz)).$$

It follows that $Tz=\{z\}$.

Thus

$$Tz=\{z\}=\{gz\} \quad\text{...} (\text{ II })$$

From (I) and (II) we have, $Sz=Tz=\{z\}=\{fz\}=\{gz\}$.

Hence z is a common fixed point of S, T, f and g .

Uniqueness of common fixed point follows easily from (2.1.1) .

Thus z is the unique common fixed point of S, T, f and g .

Similarly we can prove the theorem if (2.1.3) (b) holds.

Now we give an example which illustrates our Theorem 2.1.

Example 2.2. Let $X = [0,1]$ endowed with usual metric d.

Define $S, T : X \to B(X)$ and $f, g : X \to X$ by

$$fx = \begin{cases} \dfrac{1}{2} & \text{if } x \in [0,\frac{1}{2}], \\ \dfrac{x+1}{4} & \text{if } x \in (\frac{1}{2},1] \end{cases}, \quad gx = \begin{cases} 1-x & \text{if } x \in (0,\frac{1}{2}] \\ 0 & \text{if } x \in (\frac{1}{2},1] \cup \{0\} \end{cases},$$

$$Sx = \left\{\frac{1}{2}\right\} \text{ and } Tx = \begin{cases} \left\{\dfrac{1}{2}\right\} & \text{if } x \in [0,\frac{1}{2}] \\ \left[\dfrac{3}{8},\dfrac{1}{2}\right] & \text{if } x \in (\frac{1}{2},1] \end{cases}$$

Then $Sx = \left\{\frac{1}{2}\right\} \subseteq g(X) = [\frac{1}{2},1) \cup \{0\}$, $Tx = \left[\frac{3}{8},\frac{1}{2}\right] \not\subseteq f(X) = \left(\frac{3}{8},\frac{1}{2}\right]$ forall $x \in (\frac{1}{2},1]$.

Case(i) : If $x \in X$, $y \in [0,\frac{1}{2}]$ then, $\delta(Sx, Ty) = 0$.

Case(ii) : If $x \in X$, $y \in (\frac{1}{2},1]$ then, $\delta(Sx,Ty) = \frac{1}{8}$, $d(fx,gy) > \frac{3}{8}$.

Thus in all cases, we have

$$\delta(Sx,Ty) \le \frac{1}{3}\max\{d(fx,gy) + \delta(fx,Sx) + \delta(gy,Ty), \delta(fx,Ty), \delta(gy,Sx)\} .$$

The inequality (2.1.1) is satisfied with $\varphi(t) = \frac{2t}{3}$. Clearly ($f$, S) and (g, T) are subcompatible, since they commute at their coincidence point $x = \frac{1}{2}$.

Now for $\{x_n\} = \frac{1}{2} - \frac{1}{2n}$, we have $fx_n \to \frac{1}{2}$, $Sx_n = \left\{\frac{1}{2}\right\}$, $\frac{1}{2} \in f(X)$.

$Sx \subseteq g(X)$, $\forall\, x \in X$. Hence the condition (2.1.3)(a) is satisfied. The condition (2.1.3)(b) is not satisfied , since $Tx \not\subseteq f(X), \forall\, x \in (\frac{1}{2},1]$.

Thus all the conditions of Theorem 2.1 are satisfied and $\frac{1}{2}$ is the unique common fixed point of f, g, S and T.

Remark. In Example 2.2 , note that the maps f and g are not surjective and the sets $f(X)$ and $g(X)$ are not closed.

Corollary 2.3. Let f, g be self maps on a metric space (X, d) and let $S, T : X \to B(X)$ be set-valued maps satisfying (2.1.2), (2.1.3)(a) or (2.1.3)(b) and

$$\text{(2.3.1)} \qquad \delta(Sx,Ty) \le \varphi\left(\max\left\{\begin{matrix} d(fx,gy) + \delta(fx,Sx) + \delta(gy,Ty), \\ \delta(fx,Ty), \delta(gy,Sx) \end{matrix}\right\}\right)$$

for all $x, y \in X$, where $\varphi : [0,\infty) \to [0,\infty)$ is continuous and $\varphi(t) < t$ for all $t > 0$.

Then f, g, S and T have a unique common fixed point in X.

Proof. Follows from Theorem 2.1, by putting $\varphi(t) = t - \varphi(t)$.

Now we give the following

Theorem 2.4: Let f, g be self maps on a metric space (X, d) and let $S, T : X \to B(X)$ be set-valued maps such that

$$(2.4.1)\quad \psi(\delta(Sx,Ty)) \le \psi\left(\max\begin{Bmatrix} d(fx,gy), \delta(fx,Sx), \delta(gy,Ty), \delta(fx,Ty), \\ \delta(gy,Sx) \end{Bmatrix}\right) - \varphi\left(\max\begin{Bmatrix} d(fx,gy), \delta(fx,Sx), \delta(gy,Ty), \\ \delta(fx,Ty), \delta(gy,Sx) \end{Bmatrix}\right)$$

for all $x, y \in X$, where $\psi, \varphi : R_+ \to R_+$ are continous and $\varphi(t) > 0$ for all $t > 0$.

(2.4.2) (f, S) and (g, T) are subcompatible pairs and

(2.4.3) either the maps f and S or the maps g and T are D-maps with respect to the pair (f, g).

Then f, g, S and T have a unique common fixed point in $z \in X$ such that $Sz = Tz = \{fz\} = \{gz\} = \{z\}$.

Proof. Suppose that the maps f and S are D-maps with respect to the pair (f, g). Then there exists a sequnce $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = z$ *and* $\lim_{n\to\infty} Sx_n = \{z\}$ for some $z \in f(X) \cap g(X)$.

Hence there exists $u, v \in X$ such that $z = fu = gv$.

Putting $x = x_n$, $y = v$ in (2.4.1) we get,

$$\psi(\delta(Sx_n,Tv)) \le \psi\left(\max\begin{Bmatrix} d(fx_n,gv), \delta(fx_n,Sx_n), \delta(gv,Tv), \\ \delta(fx_n,Tv), \delta(gv,Sx_n) \end{Bmatrix}\right) - \varphi\left(\max\begin{Bmatrix} d(fx_n,gv), \delta(fx_n,Sx_n), \delta(gv,Tv), \\ \delta(fx_n,Tv), \delta(gv,Sx_n) \end{Bmatrix}\right)$$

Letting $n \to \infty$, we get

$$\psi(\delta(z,Tv)) \le \psi\left(\max\{0,0,\delta(z,Tv),\delta(z,Tv),0\}\right) - \varphi\left(\max\{0,0,\delta(z,Tv),\delta(z,Tv),0\}\right).$$
$$= \psi(\delta(z,Tv)) - \varphi(\delta(z,Tv)).$$

Hence $\varphi(\delta(z, Tv)) \le 0$ so that $Tv = \{z\}$.

Thus

$$\{gv\} = \{z\} = Tv \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\text{III})$$

Putting $x = u$, $y = v$ in (2.4.1), we have $Su = \{z\}$.

Thus

$$\{fu\} = \{z\} = Su \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{IV})$$

From (III), (IV) and (2.4.2) we have $\{fz\} = Sz$ and $\{gz\} = Tz$.

Putting $x = z$ and $y = v$ in (2.4.1), we have $fz = z$. Thus $Sz = \{fz\} = \{z\}$.

Putting $x = u$ and $y = z$ in (2.4.1), we have $gz = z$. Thus $Tz = \{gz\} = \{z\}$.

Thus z is a common fixed point of $f, g, S,$ and T and $Sz = Tz = \{fz\} = \{gz\} = \{z\}$.

Uniquness of common fixed point follows form (2.4.1).

Similarly the theorem holds when the maps g and T are D-maps with respect to (f, g).

The following example illustrates our Theorem 2.4.

Example 2.5. Let $X = [0, 1]$ endowed with usual metric d. Define $S, T : X \to B(X)$ and $f, g : X \to X$ by

$$fx = \begin{cases} \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}] \\ \frac{x+1}{4} & \text{if } x \in (\frac{1}{2}, 1] \end{cases}, \qquad gx = \begin{cases} 1-x & \text{if } x \in (0, \frac{1}{2}] \\ 0 & \text{if } x \in (\frac{1}{2}, 1] \cup \{0\} \end{cases}$$

$$Sx = \begin{cases} \left\{\frac{1}{2}\right\} & \text{if } x \in [0, \frac{1}{2}] \\ \left[\frac{7}{16}, \frac{1}{2}\right] & \text{if } x \in (\frac{1}{2}, 1] \end{cases} \quad \text{and } Tx = \begin{cases} \left\{\frac{1}{2}\right\} & \text{if } x \in [0, \frac{1}{2}] \\ \left[\frac{3}{8}, \frac{1}{2}\right] & \text{if } x \in (\frac{1}{2}, 1] \end{cases}$$

Case(i) : If $x, y \in [0, \frac{1}{2}]$ then $\delta(Sx, Ty) = 0$.

Case(ii) : If $x \in [0, \frac{1}{2}]$, $y \in (\frac{1}{2}, 1]$ then,

$$\delta(Sx, Ty) = \frac{1}{8}, \quad d(fx, gy) = \frac{1}{2}. \text{ Thus } \delta(Sx, Ty) < \frac{1}{2} d(fx, gy).$$

Case(iii) : If $x \in (\frac{1}{2}, 1]$, $y \in [0, \frac{1}{2}]$ then,

$$\delta(Sx, Ty) = \frac{1}{16}, \quad \delta(fx, Sx) = \frac{1}{8}. \text{ Thus } \delta(Sx, Ty) = \frac{1}{2} \delta(fx, Sx).$$

Case(iv) : If $x, y \in (\frac{1}{2}, 1]$, then,

$$\delta(Sx, Ty) = \frac{1}{8}, \quad d(fx, gy) = \frac{x+1}{4} > \frac{3}{8}. \text{ Thus } \delta(Sx, Ty) < \frac{1}{2} d(fx, gy).$$

Thus $\delta(Sx, Ty) \le \frac{1}{2} \max\{d(fx, gy), \delta(fx, Sx), \delta(gy, Ty), \delta(fx, Ty), \delta(gy, Sx)\}$.

The inequality (2.4.1) is satisfied with $\psi(t) = t, \quad \varphi(t) = \frac{t}{2}$. Clearly (f, S) and (g, T) are subcompatible.

Now for $\{x_n\} = \frac{1}{2} - \frac{1}{2n}$, we have $gx_n \to \frac{1}{2}, \quad Tx_n = \frac{1}{2}, \quad \frac{1}{2} \in f(X) \cap g(X)$.

Hence the maps g and T are D-maps with respect to (f, g). $\frac{1}{2}$ is the unique common fixed point of f, g, S and T.

Remark. In Example 2.5 , note that

(i) f and g are not surjective,

(ii) $f(X)$ and $g(X)$ are not closed,

(iii) $Sx \not\subseteq g(X) \ \forall x \in (\frac{1}{2}, 1], \quad Tx \not\subseteq f(X) \ \forall x \in (\frac{1}{2}, 1]$

Corollary 2.6. Theorem 2.4 holds if the inequality (2.4.1) is replaced by

$$\delta(Sx, Ty) \le \varphi\left(\max\left\{\begin{matrix} d(fx, gy), \delta(fx, Sx), \delta(gy, Ty), \\ \delta(fx, Ty), \delta(gy, Sx) \end{matrix}\right\}\right) \tag{2.6.1}$$

for all $x, y \in X$, where $\varphi : R_+ \to R_+$ is continous and $\varphi(t) < t$ for all $t > 0$.

Proof. Putting $\psi(t) = t$, $\varphi(t) = t - \varphi(t)$ in Theorem 2.4 we get the proof of the corollary.

Finally, one can easily prove the following:

Theorem 2.7. Theorem 2.4 holds if the inequality (2.4.1) is replaced by one of the following

(2.7.1) $$d(fx,gy) \geq \max\{\delta(Sx,Ty),\delta(fx,Sx),\delta(gy,Ty),\delta(fx,Ty),\delta(gy,Sx)\} + \varphi\left(\max\{\delta(Sx,Ty),\delta(fx,Sx),\delta(gy,Ty),\delta(fx,Ty),\delta(gy,Sx)\}\right).$$

for all $x, y \in X$, where $\varphi : R_+ \to R_+$ is continous and $\varphi(t) > 0$ for all $t > 0$.

(2.7.2) $$d(fx,gy) \geq \varphi\left(\max\{\delta(Sx,Ty),\delta(fx,Sx),\delta(gy,Ty),\delta(fx,Ty),\delta(gy,Sx)\}\right)$$

for all $x, y \in X$, where $\varphi : R_+ \to R_+$ is continous and $\varphi(t) > 0$ for all $t > 0$.

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