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A TRANSPLANTATION THEOREM FOR THE HANKEL TYPE TRANSFORM ON THE HARDY SPACE

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Abstract

In the present paper the transplantation operators for the Hankel type transform are considered and their boundedness on the real Hardy space is established. As its application, we have obtained the Hormander-Mihlin type multiplier theorem for the Hankel transform on the real Hardy space.

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1.Introduction

The Hankel type transform of $\mathcal{H}_{\alpha,\beta}f$ of order $\alpha-\beta$ of a function on the open half-line $(0,\infty)$ is defined by

$$\mathcal{H}_{\alpha,\beta}f(y) = \int_0^\infty f(t)\,(yt)^{\alpha+\beta} J_{\alpha-\beta}(yt)\,dt\,,\quad y>0\,,$$

where $J_{\alpha-\beta}$ is the Bessel type function of the first kind of order $\alpha-\beta$. The Bessel functions with $\alpha-\beta=-\frac{1}{2}$ and $\alpha-\beta=\frac{1}{2}$ are

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\,cosz\,,\qquad J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\,sinz$$

and the Hankel transforms $\mathcal{H}_{-\frac{1}{2}}f$ and $\mathcal{H}_{\frac{1}{2}}f$ are the cosine and sine transforms:

$$\mathcal{H}_{-\frac{1}{2}}f(y) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\,cosyt\,dt\,,\ \ \mathcal{H}_{\frac{1}{2}}f\,(y) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\,siny\,t\,dt\,.$$

It is known that for $(\alpha-\beta)\geq-\frac{1}{2}$, $\mathcal{H}_{\alpha,\beta}$ is an isometry on $L^2(0,\infty)$ (Parseval's theorem) for the Hankel type transform) and $\mathcal{H}_{\alpha,\beta}\,.\mathcal{H}_{\alpha,\beta}=I$ (The inversion formula for the Hankel type transform), and

$$\int_0^\infty f(x)\,g(x)\,dx = \int_0^\infty \mathcal{H}_{\alpha,\beta}\,f\,(x)\,\mathcal{H}_{\alpha,\beta}\,g(x)\,dx$$

for $f,g\in L^2\,(0,\infty)$ (Plancherel's theorem for the Hankel type tranform), where I is the identity operator and $L^2\,(0,\infty)$ is the Lebesgue space of functions on $(0,\infty)$ such that

$$\|f\|_2 = \left(\int_0^\infty |f(x)|^2\,dx\right)^{\frac{1}{2}} < \infty\,.$$

We shall consider the composite

$$T_{\alpha,\beta}^{a,b} = \mathcal{H}_{\alpha,\beta}\,.\mathcal{H}_{a,b}\ ,$$

which is an isometry on $L^2\,(0,\infty)$ for $(\alpha-\beta)\geq-\frac{1}{2}$, $(a-b)\ \geq\ -\frac{1}{2}$. For $f\ \in\ L^{'}\,(0,\infty)$ with $\mathcal{H}_{a,b}\,f\ \in\ L^{'}\,(0,\infty)$, $T_{\alpha,\beta}^{a,b}\,f$ has the integral representation

$$T_{\alpha,\beta}^{a,b}\,f(x) = \int_0^\infty\int_0^\infty f(t)\,(yt)^{\alpha+b}\,J_{a-b}\,(yt)\,dt\ \,(xy)^{\alpha-\beta}\ J_{\alpha-\beta}\,(xy)\,dy,\qquad x>0.$$

We call $T_{\alpha,\beta}^{a,b}$ the transplantation operator from $a-b\ to\ \alpha-\beta$. The aim of this paper is to prove that the transplantation operators $T_{\alpha,\beta}^{a,b}$ are bounded on the real Hardy space.

As an application, we shall obtain the Hormander-Mihlin type multiplier theorem for the Hankel type transform on the real Hardy space.

The main tools of our proofs are the atomic decomposition and the molecular characterization of the real Hardy space, and Schindler's integral representation [13] of $T_{\alpha,\beta}^{a,b}$. We recall the representation here. Let $T_{\alpha,\beta,a,b}$ be an operator defined by

$$T_{\alpha,\beta,a,b}\, f(x) = \lim_{\delta\to 0+}\int_{|x-y|>\delta} f(y)\, \tilde{I}_{\alpha,\beta,a,b}\,(x,y)\, dy + k(\alpha,\beta,a,b)\, f(x)\,, \tag{1.1}$$

$$k\,(\alpha,\beta,a,b) = Cos\left((\alpha-\beta-a+b)\frac{\pi}{2}\right),$$

where

$$\tilde{I}_{\alpha,\beta,a,b}\,(x,y) = K_{\alpha,\beta,a,b}\,(xy)^{\frac{1}{2}}\left(\frac{y}{x}\right)^{a-b}\frac{1}{x^2-y^2}\, F\left(\frac{a-b-\alpha+\beta}{2},\frac{\alpha-\beta+a-b}{2};3a+b\,;\frac{y^2}{x^2}\right), \tag{1.2}$$

$$= \frac{1}{2}\, K_{\alpha,\beta,a,b}\left(\frac{y}{x}\right)^{2a}\left(\frac{1}{x-y}+\frac{1}{x+y}\right)$$

$$\times\, F\left(\frac{a-b-\alpha+\beta}{2},\frac{\alpha-\beta+a-b}{2};3a+b\,;\frac{y^2}{x^2}\right),$$

$$K_{\alpha,\beta,a,b} = \frac{2\,\Gamma(\alpha-\beta+a-b+2)/2)}{\Gamma(3a+b)\,\Gamma\left((\alpha-\beta-a+b)/2\right)}\,,\qquad for\;\; 0<y<x\,,$$

and

$$\tilde{I}_{\alpha,\beta,a,b}\,(x,y) = \tilde{I}_{\alpha,\beta,a,b}\,(y,x)\,,\qquad for\;\; y>x>0.$$

Here

$$F\,(p',q';r';z) = \sum_{k=0}^{\infty}\frac{(p')_k(q')_k}{(r')_k\,k!}\,z^k\,,\qquad |z|<1\,,$$

where $(\lambda)_0 = 1, (\lambda)_k = \lambda(\lambda+1)\ldots\ldots(\lambda+k-1)$, $k \geq 1$. If $\alpha-\beta = a-b+2k$ and $k = 0,1,2\ldots.$, then $\tilde{I}_{\alpha,\beta,a,b}\;(x,y) = 0$ for $(y>x>0)$. If $k = 0,-1,-2,\ldots\ldots.$, then $\tilde{I}_{\alpha,\beta,a,b}\;(x,y) = 0$ for $x>y>0$. In these cases, $\tilde{I}_{\alpha,\beta,a,b}\;(x,y)$ have more elementary forms (See [13]).

Schindler proved that if $(\alpha-\beta),(a-b) \geq -\frac{1}{2}$, then the following (A) and (B) hold:

A. For $f \in C_c^{\infty}(0,\infty)\,, T_{\alpha,\beta}^{a,b}\, f(x) = T_{\alpha,\beta,a,b}\, f(x)\; a.e.\;\; , x>0$, where $C_c(0,\infty)$ is the space of infinitely differentiable functions of compact support in $(0,\infty)$;

B. Let $|< p < \infty$ and $-\frac{1}{p} < a < 1-\frac{1}{p}$. If $\int_0^{\infty}|f(x)|^p\, x^{p'p}\, dx < \infty$, then the value $T_{\alpha,\beta,a,b}\, f(x)$ exists for a.e. $x>0$ and

$$\int_0^{\infty}\left|T_{\alpha,\beta,a,b}\, f(x)\right|^p x^{p'p}\, dx \leq C\int_0^{\infty}|f(x)|^p\, x^{p'p} dx\;.$$

with a constant C independent of f. Guy [6] proved that the operators $T_{\alpha,\beta}^{a,b}$ initially defined on $L^2(0,\infty)$, are extendable to bounded operators on the L^p- spaces, $1<p<\infty$, and this is the first of the transplantation theorem for classical expansions. Schindler [13] showed a refined version of Guy's result by getting the explicit formula of $T_{\alpha,\beta}^{a,b}$ as we recalled above.

To consider the transplantation operators $T_{\alpha,\beta}^{a,b}$ for the case $p = 1$ is our problem, and the main result of this paper is that the operator $T_{\alpha,\beta}^{a,b}$ are bounded on the real Hardy space which gives us the Hormander-Mihlin type multiplier theorem for the Hankel type transform on the real Hardy space. There are transplantation theorems for other orthogonal expansions. Askey and Woinger [2] gave a transplantation theorem for the ultra spherical series, and Askey [1] generalized their theorem to the Jacobi series. Some transplantation theorems are in Gilbert [5] and in Muckenhoupt [12]. The Laguerre series case is in Kanjin [7]. Miyachi [10] and [11] quite recently obtained a transplantation theorem for the Jacobi series in weighted Hardy spaces.

2. Results

Let $H'(\mathbb{R})$ be the real Hardy space that is the space of the boundary functions $f(x) = \mathcal{R}\, F(x)$ of the real part $\mathcal{R}\, F(z)$ of functions $F(z)$ in the Hardy space $H'(\mathbb{R}_+^2) = \{F(z);$ analytic in $\mathbb{R}_+^2$ and $\|F\|_{H'(\mathbb{R}_+^2)} = Sup_{t>0}\int_{-\infty}^{\infty}|F(x+it)|\, dx < \infty\}$ on the upper half plane $\mathbb{R}_+^2 = \{z = x + it \; ; \; t > 0\}$, with the norm $\|f\|_{H'(\mathbb{R})} = \|F\|_{H'(\mathbb{R}_+^2)}$. We shall work on the space $H'(0,\infty)$ defined by

$$H'(0,\infty) = \{h\,|_{(0,\infty)} \; ; h \in H'(\mathbb{R}),\; Supph \subset [0,\infty)\}.$$

where $[0,\infty)$ is the closed half line and we endow the space with the norm

$$\|f\|_{H'(0,\infty)} = \|h\|_{H'(\mathbb{R})}, \qquad where\;\; h \in H'(\mathbb{R}),$$
$$Supph \subset [0,\infty)\; and\; f = h\,|_{(0,\infty)}.$$

We remark that

$$H'(0,\infty) = \{h\,|_{(0,\infty)} \; ; \; h \in H'(\mathbb{R}),\;\; even\}$$

and $c_1\,\|h\|_{H'(\mathbb{R})} \le \|f\|_{H'(0,\infty)} \le c_2\,\|h\|_{H'(\mathbb{R})}$ with positive constants $c_1\; and\; c_2$, $where\; f = h\,|_{(0,\infty)}\; and\; h \in H'(\mathbb{R})$ is even. This fact is in [4, chapter III, Lemma 7.40].

Our main theorem is as follows:

Theorem

(i) Let $(\alpha - \beta) \ge -\frac{1}{2}\; and\; (a-b) > -\frac{1}{2}$. Then $T_{\alpha,\beta}^{a,b}$, initially defined on $H'(0,\infty) \cap L^2(0,\infty)$, is uniquely extended to a bounded operator on $H'(0,\infty)$ and if we still denote it by $T_{\alpha,\beta}^{a,b}$, then

$$\left\|T_{\alpha,\beta}^{a,b} f\right\|_{H'(0,\infty)} \le C\,\|f\|_{H'(0,\infty)}, \qquad for\, f \in H'(0,\infty)$$

with a constant C depending only on $\alpha - \beta$ and $a - b$.

(ii) If $(\alpha - \beta) \ge -\frac{1}{2}$, then $T_{\alpha,\beta}^{-\frac{1}{2}}$ is uniquely extended to a bounded operator from $H'(0,\infty)\; to\; L'(0,\infty)$, that is

$$\left\|T_{\alpha,\beta}^{-\frac{1}{2}} f\right\|_{L'(0,\infty)} \le C\,\|f\|_{H'(0,\infty)}, \qquad for\;\; f \in H'(0,\infty)$$

with a constant C depending only on $\alpha - \beta$ and $a - b$.

As an application of our theorem we deal with the Hormander-Mihlin type multiplier theorem for the Hankel type transform. Let $(\alpha - \beta) \ge -\frac{1}{2}$ and $\Phi \in L^\infty(0,\infty)$. We define a Hankel multiplier operator $\mathcal{M}_\Phi^{\alpha,\beta}$ with multiplier ϕ by

$$\mathcal{M}_\phi^{\alpha,\beta} f = \mathcal{H}_{\alpha,\beta}\left(\phi\,\mathcal{H}_{\alpha,\beta}(f)\right), \qquad for\, f \in L^2(0,\infty).$$

Since $\mathcal{H}_{\alpha,\beta}$ is an isometry on $L^2(0,\infty)$, the multiplier operator $\mathcal{M}_\phi^{\alpha,\beta}$ is a bounded operator on $L^2(0,\infty)$ with the operator norm $\|\phi\|_\infty$. We also define a Fourier multiplier operator $\mathcal{M}_m$ with multiplier $m \in L^\infty(\mathbb{R})$ by

$$\mathcal{M}_m h = \mathcal{F}^{-1}\left(m\,\mathcal{F}(h)\right), \qquad for\;\; h \in L^2(\mathbb{R}),\;\; where\; \mathcal{F}\; and\; \mathcal{F}^{-1}$$

are the Fourier transform and the inverse Fourier transform, respectively:

$$\mathcal{F}(h)(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} h(x)\,e^{-ix\xi}\,dx\,, \;\; \mathcal{F}^{-1}(g)(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g(\xi)\,e^{ix\xi}\,d\xi$$

The Hormander-Mihlin multiplier theorem for $H'(\mathbb{R})$ says that, if m with $\|m\|_{L^\infty(\mathbb{R})} \le A$ satisfies the condition

$$\left(\frac{1}{R}\int_{R<|\xi|<2R}\left|\frac{dm(\xi)}{d\xi}\right|^2 d\xi\right)^{\frac{1}{2}} \le A\,R^{-1} \;\; for\; R > 0, \qquad where \tag{2.1}$$

A is independent of R, then the Fourier multiplier operator $\mathcal{M}_m$ initially defined on $H'(\mathbb{R}) \cap L^2(\mathbb{R})$ is uniquely extended to a bounded operator on $H'(\mathbb{R})$. If we still denote it by $\mathcal{M}_m$, then $\|\mathcal{M}_m h\|_{H'(\mathbb{R})} \le C\,A\,\|h\|_{H'(\mathbb{R})}\;\; for\;\; h \in H'(\mathbb{R})$ with C independent of h and m (see [4, Chapter III, Theorem 7.30]). We may refer to [14, Chapter IV, §3, §6] and [4, Chapter II, Theorem 6.3] for the $L^p -$ space case.

Corollary

Let $(\alpha - \beta) \ge -\frac{1}{2}$. Suppose that ϕ with $\|\phi\|_{L^\infty(0,\infty)} \le A$ satisfies the condition

$$\left(\frac{1}{R}\int_{R<|y|<2R}\left|\frac{d\phi(y)}{dy}\right|^2 dy\right)^{\frac{1}{2}} \le A\,R^{-1} \tag{2.2}$$

for $R > 0$, where A is independent of R. Then the Hankel multiplier operator $\mathcal{M}_\phi^{\alpha,\beta}$ initially defined on $H^1(0,\infty) \cap L^2(0,\infty)$ is uniquely extended to a bounded operator from $H^1(0,\infty)\; to\; L^1(0,\infty)$. If we also denote it by $\mathcal{M}_\phi^{\alpha,\beta}$ then

$$\left\|M_\phi^{\alpha,\beta} f\right\|_{L^1(0,\infty)} \le C\,A\,\|f\|_{H'(0,\infty)}\,, \qquad for\;\; f \in H^1(0,\infty)\; with\; C$$

independent of $f\; and\; \phi$.

The corollary is deduced from the theorem as follows: Let $\phi \in L^\infty(0,\infty)$ satisfy the condition (2.2) and let $f \in H^1(0,\infty) \cap C_c^\infty(0,\infty)$.
We extend ϕ *and* f to the functions on $\mathbb{R}$, as even functions and we denote them by ϕ_e *and* f_e. Since the function ϕ_e satisfies the condition (2.1), the Fourier multiplier operator $\mathcal{M}_{\phi_e}$ is a bounded operator on $H^1(\mathbb{R})$. Since $H_{-\frac{1}{2}} f(y) = \mathcal{F} f_e(y),\ y > 0$, we see that $\mathcal{M}_{\phi_e} f_e(x) = \mathcal{M}_\phi^{-\frac{1}{2}} f(x),\ x > 0$. Further $\mathcal{M}_{\phi_e} f_e$ is an even function. Thus $\mathcal{M}_\phi^{-\frac{1}{2}}$ has a unique bounded extension on $H^1(0,\infty)$. The inequality $\|g\|_{L^1(0,\infty)} \le \|g\|_{H^1(0,\infty)}$ holds, and so $\mathcal{M}_\phi^{-\frac{1}{2}}$ is uniquely extended to a bounded operator from $H^1(0,\infty)$ to $L^1(0,\infty)$. Let $(\alpha-\beta) > -\frac{1}{2}$. It follows from the theorem that $T^{\alpha,\beta}_{-\frac{1}{2}}$ is a bounded operator on $H^1(0,\infty)$ and $T_{\alpha,\beta}^{-\frac{1}{2}}$ is a bounded operator from $H^1(0,\infty)$ to $L^1(0,\infty)$. Therefore, the identity $\mathcal{M}_\phi^{\alpha,\beta} = T_{\alpha,\beta}^{-\frac{1}{2}} \mathcal{M}_\phi^{-\frac{1}{2}} T^{\alpha,\beta}_{-\frac{1}{2}}$ on $L^2(0,\infty)$ implies the corollary.

Remark : Let $\alpha - \beta > -\frac{1}{2}$. Assume that $\mathcal{M}_\phi^{\alpha,\beta}$ is a bounded operator on $H^1(0,\infty)$. Then $\phi = 0$ if we assume additionally that ϕ satisfies $\phi = H_{-\frac{1}{2}}\Phi$ for some $\Phi \in L^1(0,\infty)$. For, we first note that $\mathcal{M}_\phi^{\frac{1}{2}}$ is a bounded operator on $H^1(0,\infty)$ by the identity $\mathcal{M}_\phi^{\frac{1}{2}} = T^{\alpha,\beta}_{\frac{1}{2}} \mathcal{M}_\phi^{\alpha,\beta} T^{\frac{1}{2}}_{\alpha,\beta}$ and the theorem. Let $f \in H^1(0,\infty) \cap C_c^\infty(0,\infty)$. Since $\mathcal{M}_\phi^{\frac{1}{2}} f \in H^1(0,\infty)$, $\mathcal{M}_\phi^{\frac{1}{2}} f$ has the vanishing mean property:

$$\int_0^\infty \mathcal{M}_\phi^{\frac{1}{2}} f(x)\,dx = 0.$$

We extend ϕ and Φ to the even functions on $\mathbb{R}$, and denote them by ϕ_e and Φ_e. We note that $\phi_e = \mathcal{F}\,\Phi_e$. Further, we extend f to the odd function on $\mathbb{R}$, which is denoted by f_o. Since $-i\,\mathcal{H}_{\frac{1}{2}} f(y) = \mathcal{F} f_o(y),\ y > 0$, we see that $\mathcal{M}_\phi^{\frac{1}{2}} f(x) = \mathcal{M}_{\phi e} f_o(x)$, $x > 0$. The identity $\mathcal{M}_{\phi e} f_o = \Phi_e * f_o$ holds. Therefore, we have

$$\begin{aligned} 0 &= \int_0^\infty \int_{-\infty}^\infty \Phi_e(y)\, f_o(x-y)\,dy\,dx \\ &= \int_{-\infty}^\infty \Phi_e(y) \int_0^\infty f_o(x-y)\,dx\,dy \\ &= \int_0^\infty \Phi_e(y) \int_{-y}^\infty f_o(u)\,du\,dy + \int_{-\infty}^0 \Phi_e(y) \int_{-y}^\infty f_o\,du\,dy \\ &= -2\int_0^\infty \Phi(y) \int_0^y f(u)\,dudy = -2\int_0^\infty f(u) \int_u^\infty \Phi(y)\,dy\,du, \end{aligned}$$

that is $\int_0^\infty f(u)\,\Phi(y)\,dy\,du = 0$ for all $f \in H^1(0,\infty) \cap C_c^\infty(0,\infty)$, which implies that $\int_u^\infty \Phi(y)\,dy$ is a constant function in u, and so $\Phi(y) = 0$ for $a.e.\, y > 0$. We conclude $\phi = 0$.

We conjecture that without the additional condition the above statement holds, that is if $\mathcal{M}_\phi^{\alpha,\beta}$ with $\phi \in L^\infty(0,\infty)$ is bounded operator on $H^1(0,\infty)$, then ϕ is constant, where $(\alpha-\beta) > -\frac{1}{2}$.

The theorem will be proved in the next section. The atomic decomposition and the molecular characterization of the real Hardy space will play important role in our proof. A real valued function a is called an atom centered at c if
(i) $a(x)$ is supported in an interval $[c - \frac{h}{2},\ c + \frac{h}{2}]$
(ii) $\|a\|_2 \le h^{-\frac{1}{2}}$, *and* (iii) $\int_{\mathbb{R}} a(x)dx = 0$. The space $H^1(\mathbb{R})$ is characterized in terms of atoms: $f \in H^1(\mathbb{R})$ if and only if

$$f = \sum_{j=0}^\infty \lambda_j\, a_j\,,$$

where each a_j is an atom and

$$\sum_{j=0}^{\infty} |\lambda_j| < \infty.$$

Further, the $H^1 - norm\ \|f\|_{H^1(\mathbb{R})}$ is equivalent to

$$inf \sum_{j=0}^{\infty} |\lambda_j| ,$$

the infimum being taken over all decompositions, and the series

$$\sum_{j=0}^{\infty} \lambda_j\, a_j$$

converges in $H^1 - norm$.

We deal with the functions $f\ \in H^1(0,\infty)$. These functions are also characterized as follows (See [4, chapter III, Lemma 7.40]) : $f\ \in H^1(0,\infty)$ if and only if

$$f\ = \sum_{j=0}^{\infty} \lambda_j\, a_j\ ,$$

where each a_j is an atom with $Supp\ a_j\ \subset [0,\infty)$ and

$$\sum_{j=0}^{\infty} |\lambda_j| < \infty.$$

Moreover, the norm $\|f\|_{L^1(0,\infty)}$ is equivalent to

$$inf \sum_{j=0}^{\infty} |\lambda_j| ,$$

the infimum being taken over all such decompositions. By this decomposition we see that $L^1(0,\infty)\ \cap\ L^2\ (0,\infty)$ is dense in $H^1(0,\infty)$.

We call a real-valued function M a molecule centered at c if M satisfies the following conditions:

(i) $N\ (M) =\ M^{\frac{1}{2}}_{L^2(\mathbb{R})}\ |\cdot\ -c|\ M^{\frac{1}{2}}_{L^2(\mathbb{R})} <\ \infty$;

(ii) $\int_{\mathbb{R}}\ M\ (x)\ dx\ = 0$.

We recall $N(M)$ the molecular norm of $M(x)$. The molecular characterization asserts that if $f\ =\ \Sigma_j\ M_j$ with molecules M_j and $\Sigma_j\ N(M_j) < \infty$, then $f\ \in H^1\ (\mathbb{R})$ and $\|f\|_{H^1(\mathbb{R})}\ \leq C\ \Sigma_j\ N(M_j)$ with an absolute constant C. For the atomic decomposition and the molecular characterization, we may refer to $[4,\ III]$.

3. Proofs: The theorem will be proved by the following two lemmas:

Lemma 3.1: If $(a-b) > -\frac{1}{2}$, then $T^{5a,3\beta}_{a,b}$ and $T^{a,b}_{5a,3b}$ are uniquely extended to bounded operators on $H^1(0,\infty)$, that is

$$\left\|T^{5a,3b}_{a,b} f\right\|_{H^1(0,\infty)}\ \leq\ C\ \|f\|_{H^1(0,\infty)}\ ,$$

$$\left\|T^{a,b}_{5a,3b} f\right\|_{H^1(0,\infty)}\ \leq C\ \|f\|_{H^1(0,\infty)}$$

for $f\ \in H^1(0,\infty)$ with a constant C depending only on $a-b$.

Lemma 3.2: (i) If $(a-b) > -\frac{1}{2}$ and $(a-b) > -\frac{1}{2}$, then $T^{a,b}_{\alpha,\beta}$ is uniquely extended to a bounded operator on $L^1(0,\infty)$, that is

$$\left\|T^{a,b}_{\alpha,\beta} f\right\|_{H^1(0,\infty)}\ \leq C\ \|f\|_{H^1(0,\infty)}\ ,\qquad f\ \in H^1(0,\infty)$$

with a constant c depending only on $\alpha-\beta$ and $a-b$.

(ii) If $(\alpha-\beta) \geq -\frac{1}{2}$, then $T^{-\frac{1}{2}}_{\alpha,\beta}$ is uniquely extended to a bounded operator from $H^1(0,\infty)$ to $L^1(0,\infty)$, that is

$\left\|T^{-\frac{1}{2}}_{\alpha,\beta}\right\|_{L^1(0,\infty)}\ \leq C\ \|f\|_{H^1(0,\infty)}\ ,\ for\ \ f \in H^1(0,\infty)$ with a constant c depending only on $\alpha-\beta$.

We see here that the theorem is deduced from these lemmas. We first note that the identity $T^{\tau}_{\alpha,\beta}\ T^{a,b}_{\tau}\ =\ T^{a,b}_{\alpha,\beta}$ on $L^2(0,\infty)$ holds, since

$$T^{\tau}_{\alpha,\beta}\ T^{a,b}_{\tau}\ =\ \mathcal{H}_{\alpha,\beta}\ \mathcal{H}_{\tau}\ \mathcal{H}_{\tau}\ \mathcal{H}_{a,b}\ =\ \mathcal{H}_{\alpha,\beta}\ \mathcal{H}_{a,b}\ =\ T^{a,b}_{\alpha,\beta}.$$

Let us prove the part (i) of the theorem. Let $(\alpha-\beta)\geq-\frac{1}{2}$ and $(\alpha-\beta)>-\frac{1}{2}$. It follows from Lemma 3.1 that $T^{a,b}_{5a,3b}$, initially defined on $H^1(0,\infty)\cap L^2(0,\infty)$ is uniquely extended to a bounded operator on $H^1(0,\infty)$. Since $5a+3b\geq\frac{1}{2}$, it follows from the part (i) of Lemma 3.2 that the operator $T^{5a,3b}_{a,b}$ is uniquely extended to a bounded operator on $H^1(0,\infty)$. Because of the fact

$$T^{a,b}_{\alpha,\beta}=T^{5a,3b}_{\alpha,\beta}\,T^{a,b}_{5a,3b}$$

on $H^1(0,\infty)\cap L^2(0,\infty)$, we see that $T^{a,b}_{\alpha,\beta}$ has a unique bounded extension on $H^1(0,\infty)$. The part (ii) of the theorem is the part (ii) of Lemma 3.2 itself.

Now we turn to the proof of Lemma 3.1 :

Let $(\alpha-\beta)>-\frac{1}{2}$, and put

$$U^{(a-b)}f(x)=\int_x^\infty\left(\frac{x}{t}\right)^{2a}f(t)\,\frac{dt}{t}\,,\qquad S^{(a-b)}f(x)=\frac{1}{x}\int_0^x\left(\frac{x}{t}\right)^{2a}f(t)\,dt$$

for $x>0$. Then we see that

$$T^{5a,3b}_{a,b}f=2(3a+b)\,U^{(a-b)}f-f\,,\qquad T^{a,b}_{5a,3b}f=2(3a+b)\,S^{(a-b)}f-f$$

for $f\in L^2(0,\infty)$ by [13, p 383, line 5 from below and p.381, line 8 from below]. In [8, Proposition], we proved that $U^{(a-b)}$ *and* $S^{(a-b)}$ are extended to bounded operators on $H^1(0,\infty)$ for $(\alpha-\beta)>-\frac{1}{2}$ and thus, $T^{5a,3b}_{a,b}$ *and* $T^{a,b}_{5a,3b}$ have the same boundedness, which is Lemma 3.1.

Lemma 3.2 will be reduced to the following Lemma 3.3 and Lemma 3.4.

Lemma 3.3

Assume that $(\alpha-\beta,a-b)$ satisfies $(\alpha-\beta)\geq-\frac{1}{2}$, $(\alpha-\beta)\geq\frac{1}{2}$ or $(\alpha-\beta)\geq-\frac{1}{2}$, $a-b=-\frac{1}{2}$. Let be an atom centered at c with Suppd $\subset[0,\infty)$, and we regard $T^{a,b}_{\alpha,\beta}\,d$ as $T^{a,b}_{\alpha,\beta}\,d(x)=0$ *for* $x\leq0$. Then, there exists a constant C depending only on $\alpha-\beta$ and $a-b$ such that

$$N\left(T^{a,b}_{\alpha,\beta}\,d\right)=\left\|T^{a,b}_{\alpha,\beta}\,d\right\|_2^{\frac{1}{2}}\left\||\cdot-c|\,T^{a,b}_{\alpha,\beta}\,d\right\|_2^{\frac{1}{2}}\leq C \tag{3.1}$$

Lemma 3.4

Let $(\alpha-\beta)\geq-\frac{1}{2}$ *and* $(a-b)\geq\frac{1}{2}$. Then, $\int_0^\infty T^{a,b}_{\alpha,\beta}\,d(x)\,dx=0$ for every atom d with Suppd C $[0,\infty)$.

We show first that Lemma 3.2 is obtained by Lemma 3.3 and Lemma 3.4. Let $f\in H^1(0,\infty)\cap L^2(0,\infty)$.

Let

$$f=\sum_{j=0}^\infty\lambda_j\,a_j$$

be an atomic decomposition of f such that

$$\sum_{j=0}^\infty|\lambda_j|\leq C\,\|f\|_{H'(0,\infty)}\,,$$

where C is independent of f. To prove Lemma 3.2, we shall first show that

$$T^{a,b}_{\alpha,\beta}\,f(x)=\textstyle\sum_{j=0}^\infty\lambda_j\,T^{a,b}_{\alpha,\beta}\,a_j(x)\ \ a.e.\ x>0\,, \tag{3.2}$$

for $(\alpha-\beta)\geq-\frac{1}{2}$, $(\alpha-\beta)\geq\frac{1}{2}$ or $\alpha-\beta\geq-\frac{1}{2}$, $a-b=-\frac{1}{2}$. Let $g\in C_c^\infty(0,\infty)$. Then we have

$$\int_0^\infty T^{a,b}_{\alpha,\beta}\,f(x)\,g(x)\,dx=\int_0^\infty\mathcal{H}_{\alpha,\beta}\,\mathcal{H}_{a,b}\,f(x)\,g(x)\,dx.$$

$$=\int_0^\infty f(x)\,\mathcal{H}_{a,b}\,\mathcal{H}_{\alpha,\beta}\,g(x)\,dx.$$

by Planchevel's theorem and the inversion formula. The inequality

$\left|\mathcal{H}_{a,b}\,\mathcal{H}_{\alpha,\beta}\,g(x)\right|\leq C\left\|\mathcal{H}_{\alpha,\beta}g\right\|_{L^1(0,\infty)}$ holds, and $\left\|\mathcal{H}_{\alpha,\beta}g\right\|_{L^1(0,\infty)}<\infty$.

Since $g\in C_c^\infty(0,\infty)$. For every atom a_j , we have $\|a_j\|_{L^1(0,\infty)}\leq1$.

Thus we have

$$\int_0^\infty f(x)\,\mathcal{H}_{a,b}\,\mathcal{H}_{\alpha,\beta}\,g(x)\,dx=\int_0^\infty\sum_{j=0}^\infty\lambda_j\,a_j(x)\,\mathcal{H}_{a,b}\,\mathcal{H}_{\alpha,\beta}\,g(x)\,dx$$

$$= \sum_{j=0}^{\infty} \lambda_j \int_0^{\infty} a_j\,(x)\ \mathcal{H}_{a,b}\ \mathcal{H}_{\alpha,\beta}\, g(x)\, dx$$
$$= \sum_{j=0}^{\infty} \lambda_j \int_0^{\infty} \mathcal{H}_{\alpha,\beta}\ \mathcal{H}_{a,b}\, a_j\,(x)\, g\,(x)\, dx.$$

We remark that the inequality $\|\Psi\|_{L^1(0,\infty)} \le 2^{\frac{3}{2}} N\,(\Psi)$ holds. $(cf.\,[4,\ Chapter\ III, Lemma\ 7.11])$. It follows from Lemma 3.3 that

$$\|\,\mathcal{H}_{\alpha,\beta}\mathcal{H}_{a,b}\, a_j\,\|_{L^1(0,\infty)} = \left\|T_{\alpha,\beta}^{a,b}\, a_j\right\|_{L^1(0,\infty)} \le C\, N\left(T_{\alpha,\beta}^{a,b}\, a_j\right) \le C.$$

Here and below, C denotes a positive constant which may differ at each different occurrence. Thus, the last sum is equal to

$$\int_0^{\infty} \sum_{j=0}^{\infty} \lambda_j\ \ \mathcal{H}_{\alpha,\beta}\, \mathcal{H}_{a,b}\, a_j\,(x) g\,(x)\, dx,$$

which leads to

$$\int_0^{\infty} T_{\alpha,\beta}^{a,b}\, f(x)\, g(x)\, dx = \int_0^{\infty} \sum_{j=0}^{\infty} \lambda_j\ T_{\alpha,\beta}^{a,b}\, a_j\,(x)\, g(x)\, dx$$

for all $g\ \in C_c^{\infty}(0,\infty)$, and we get (3.2).

Because of (3.2), we have

$$\left\|T_{\alpha,\beta}^{a,b}\, f\right\|_{H^1(0,\infty)} \le C \sum_{j=0}^{\infty} N\left(\lambda_j\right)\, T_{\alpha,\beta}^{a,b}\, a_j) \le C \sum_{j=0}^{\infty} \left|\lambda_j\right| N\left(T_{\alpha,\beta}^{a,b}\, a_j\right)$$
$$\le C \sum_{j=0}^{\infty} \left|\lambda_j\right| \le C\, \|f\|_{H^1(0,\infty)}$$

for $(\alpha-\beta) \ge -\frac{1}{2}$, $(a-b) \ge \frac{1}{2}$ by Lemma 3.3, Lemma 3.4 and the molecular characterization. If $(\alpha-\beta) \ge -\frac{1}{2}$, $a-b = -\frac{1}{2}$, $then$

$$\left\|T_{\alpha,\beta}^{a,b}\, f\right\|_{L^1(0,\infty)} \le \sum_{j=0}^{\infty} \left|\lambda_j\right|\, \left\|T_{\alpha,\beta}^{a,b}\, a_j\right\|_{L^1(0,\infty)} \le C \sum_{j=0}^{\infty} \left|\lambda_j\right|\, N\left(T_{\alpha,\beta}^{a,b}\, a_j\right)$$
$$\le \sum_{j=0}^{\infty} \left|\lambda_j\right| \le C\, \|f\|_{H^1(0,\infty)}\,.$$

These inequalities allow us to use the standard density argument, and we obtain Lemma 3.2.

We now come to the proofs of Lemma 3.3 and Lemma 3.4

Proof of Lemma 3.3

Let a' be an atom centred at a with Suppa $\subset [0,\infty)$. Let $Q = \left[c - h/2,\ \ c + {}^{h}\!/_{2}\right] \subset [0{,}\infty)$ be the smallest interval containing Suppa'. Since $T_{\alpha,\beta}^{a,b}$ is an isometry on $L^2(0,\infty)$, it follows that

$$\left\|T_{\alpha,\beta}^{a,b}\, a'\right\|_2 = \|a'\|_2 \le h^{-\frac{1}{2}} \tag{3.3}$$

To prove (3.1), it is enough to show that

$$\left\||\cdot\ - c|\, T_{\alpha,\beta}^{a,b}\, a'\right\|_2 \le C\, h^{\frac{1}{2}}.$$

We put $\bar{Q} = [c-h,\ c+h]$. we write

$$\left\||\cdot\ - c|\, T_{\alpha,\beta}^{a,b}\, a'\right\|_2^2 = \left\{ \int_{(0,\infty\,\cap\,\bar{Q}\,)} + \int_{(0,\infty)-\bar{Q}} \right\} |x-c|^2 \left|T_{\alpha,\beta}^{a,b}\, a'(x)\right|^2 dx$$
$$= V_1 + V_2\,(say).$$

For V_1, we have by (3.3),

$$V_1 \le h^2 \left\|T_{\alpha,\beta}^{a,b}\, a'\right\|_2^2 \le h\,.$$

An essential part of the proof is to show $V_2 \leq c\ h$. By Schindler's result (A) and (B), we see that $T_{\alpha,\beta}^{a,b} = T_{\alpha,\beta,a,b}$ on $L^2(0,\infty)$. Thus Schinder's integral representation (1.1) leads us to

$$T_{\alpha,\beta}^{a,b}\ a'(x) = \lim_{\delta\to 0+} \int\limits_{|x-y|>\delta} a'(y)\ \tilde{I}\ (x,y)\ dy + k(\alpha-\beta, a-b)\ a'\ (x)\ a.e.\ x > 0\ ,$$

where we put

$\tilde{I}\ (x,y) = \tilde{\mathrm{I}}_{\alpha,\beta,a,b}(x,y)$ for simplicity. For $x \in (0,\infty) - \bar{Q}$, we have

$$T_{\alpha,\beta}^{a,b}\ a(x) = \int\limits_{Q} a'(y)\ \tilde{I}\ (x,y)\ dy,$$

and thus,

$$V_2 = \int\limits_{(0,\infty)-\bar{Q}} |x-c|^2 \left|\int\limits_{Q} a'(y)\ \tilde{I}\ (x,y)\ dy\right|^2 dx\ .$$

The Taylor expansion of $\tilde{I}\ (x,y)$ in y at c and the cancellation property of atoms imply

$$\int\limits_{Q} a'(y)\ \tilde{I}\ \ (x,y)\ dy = \int\limits_{Q} a'(y)\ \frac{\partial\tilde{I}}{\partial y}\ \big(x, c+\theta(y-c)\big)\ (y-c)\ dy,\quad 0<\theta<1.$$

If we show

$\left|\frac{\partial\tilde{I}}{\partial y}\ (x,\xi)\right| \leq \frac{C}{|x-c|^2}\ ,\xi = c+\theta(y-\epsilon),\ 0<\theta<1,\ y \in Q\ ,$ (3.4)

$x \in (0,\infty) - \bar{Q}$ with C depending only on $\alpha-\beta$ and $a-b$, then

$$\left|\int\limits_{Q} a'(y)\tilde{I}\ (x,y)\ dy\right| \leq \frac{C}{|x-c|^2} \int\limits_{Q} |a'(y)|\ |y-c|\ dy$$

$$\leq \frac{C}{|x-c|^2}\ \|a\|_2\ h^{3/2} \leq \frac{C}{|x-c|^2}\ h,$$

which leads to the desired inequality

$$V_2 \leq C\ h^2 \int\limits_{(0,\infty)-\bar{Q}} \frac{dx}{|x-c|^2} \leq C\ h.$$

The rest of the proof is devoted to proving (3.4). We divide the matter into two cases;

Case I: $c+h < x\ and\ y \in Q$; and **Case II:** $0 < x < c-h\ and\ y \in Q$

We begin with case I. Since $0 < y < x$, it follows from (1.2) that

$$\frac{\partial\tilde{I}}{\partial y}\ (x,y) = 2^{-1}\ K_{\alpha,\beta,a,b}\ \{W_1^+\ (x,y) + W_2^+\ (x,y) + W_3^+\ (x,y)\},$$

where

$$W_1^+\ (x,y) = (2a)\ \left(\frac{y}{x}\right)^{-2b} \frac{1}{x}\left(\frac{1}{x-y}+\frac{1}{x+y}\right) F\left(\frac{a-b-\alpha+\beta}{2}, \frac{a-b+\alpha-\beta}{2}; 3a+b\ ; \frac{y^2}{x^2}\right),$$

$$W_2^+(x,y) = \left(\frac{y}{x}\right)^{2a} \left(\frac{1}{(x-y)^2}+\frac{-1}{(x+y)^2}\right) F\left(\frac{a-b-\alpha+\beta}{2}, \frac{a-b+\alpha-\beta}{2}; 3a+b; \frac{y^2}{x^2}\right),$$

and

$$W_3^+\ (x,y) = \left(\frac{y}{x}\right)^{2a} \left(\frac{1}{x-y}+\frac{1}{x+y}\right)\frac{\partial}{\partial y}\left\{F\left(\frac{a-b-\alpha+\beta}{2}, \frac{a-b+\alpha-\beta}{2}; 3a+b; \frac{y^2}{x^2}\right)\right\}$$

$$= \frac{(a-b)^2-(\alpha-\beta)^2}{2(3a+b)}\left(\frac{y}{x}\right)^{4a+2b} \frac{1}{x}\left(\frac{1}{x-y}+\frac{1}{x+y}\right)$$

$$F\left(\frac{a-b-\alpha+\beta+2}{2}, \frac{a-b+\alpha-\beta+2}{2}\ ; 5a+3b\ ; \frac{y^2}{x^2}\right)$$

from the formula $\frac{d}{dz}\ F\ (p',q';r';z) = (p'q'/r')\ F\ (p'+1,\ q'+1;\ r'+1\ ;\ z)$.

We shall show

$$\left|W_j^+ (x,\xi)\right| \le \frac{C}{|x-c|^2}\,, \qquad j = 1,2,3\cdot$$

with C depending only on $\alpha-\beta$ and $a-b$. (3.5)

Since

$$\lim_{z\to 1} F\,(p',q';r';z) = \frac{F(r')\,\Gamma\,(r'-p'-q')}{\Gamma(r'-p')\,\Gamma(r'-q')}$$

for $\mathcal{R}\,(r'-p'-q') > 0$ $(f.[9,\ (9.3.4)])$, it follows from

$$3a+b-(a-b-\alpha+\beta)/2\ -(a-b+\alpha-\beta)/2\ = 2(a+b) = 1$$

and $\xi < x$ that

$$\left|F\left(\frac{a-b-\alpha+\beta}{2},\frac{a-b+\alpha-\beta}{2};3a+b\,;\frac{\xi^2}{x^2}\right)\right| \le C$$

for $0 < y < x$ with a constant C depending only on $\alpha-\beta,\ a-b$.

We see that $(\xi/x)^{-2b}\ \le 1$ for $0 < y < x$ when $-2b\ \ge 0$, and that

$$|W_1^+(x,\xi)|\ \le C\,\frac{1}{x}\left(\frac{1}{|x-\xi|}+\frac{1}{x+\xi}\right).$$

Since $\xi\ \in\ Q$ and $c+h\ < x$, it follows that $|x-\xi| \ge\ |x-c|/2$.

Also $x+\xi > x\ >\ |x-c|$. These imply the inequality (3.5) with $j = 1$.

We note that the term W_1^+ does not appear in

$$\frac{\partial\tilde{I}}{\partial y}$$

when $a-b = -\frac{1}{2}$.

For $W_2^+(x,\xi)$, in a similar way, we have

$$|W_2^+(x,\xi)|\ \le C\left(\frac{1}{|x-\xi|^2}+\frac{1}{(x+\xi)^2}\right) \le \frac{C}{|x-c|^2}$$

for $a-b \ge\ -\frac{1}{2}$, which is the inequality (3.5) with $j = 2$.

To estimate $W_3^+(x,\xi)$, we use the formula $(cf.[9,9.2.6])$:

$$r'(1-z)F(p',q';r'z) - r'F(p'-1,q';r';z) + (r'-q')z\,F(p',q';r'+1;z) = 0.$$

The substitution $p' = (a-b-\alpha+\beta+2)/2$, $q' = (a-b+\alpha-\beta+2)/2$,

$$r' = 5a+3b, \qquad z = y^2/x^2\ gives$$

$$F\left(\frac{a-b-\alpha+\beta+2}{2},\frac{a-b+\alpha-\beta+2}{2};5a+3b\,;\frac{y^2}{x^2}\right)$$

$$= \frac{x^2}{x^2-y^2}\,F_1 - \frac{a-b-\alpha+\beta+2}{2(5a+3b)}\,\frac{y^2}{x^2-y^2}\,F_2\,,$$

where

$$F_1 = F\left(\frac{a-b-\alpha+\beta}{2},\frac{a-b+\alpha-\beta+2}{2};5a+3b\,;\frac{y^2}{x^2}\right),$$

$$F_2 = F\left(\frac{a-b-\alpha+\beta+2}{2},\frac{a-b+\alpha-\beta+2}{2};7a+5b\,;\frac{y^2}{x^2}\right).$$

This implies that

$$W_3^+(x,\xi) = c_{\alpha,\beta,a,b}\left(\frac{\xi}{x}\right)^{4a+2b}\left(\frac{1}{x-\xi}+\frac{1}{x+\xi}\right)^2 F_1\,|_{y=\xi}$$

$$+\,c'_{\alpha,\beta,a,b}\left(\frac{\xi}{x}\right)^{6a+4b}\left(\frac{1}{(x-\xi)^2}-\frac{1}{(x+\xi)^2}\right)F_2|_{y=\xi}\ ,$$

where $c_{\alpha,\beta,a,b}$ and $c'_{\alpha,\beta,a,b}$ are some constant depending only on $\alpha-\beta$ and $a-b$. We note that $|F_1|,|F_2|\ \le C$ for $0 < y < x$ since $5a+3b-(a-b-\alpha+\beta)/2-\ (a-b+\alpha-\beta+2)/2\ = 4(a+b)-1 = 2-1 = 1$ and $7a+5b-(a-b-\alpha+\beta+2)/2\ -(a-b+\alpha-\beta+2)/2\ = 6(a+b)-2 = 3-2 = 1$.

Thus in the same way as in the above cases, we have the inequality (3.5) with $j = 3$, which completes Case I. Now we turn to Case II. It follows from $0 < x < y$ that

$$\frac{\partial \tilde{I}}{\partial y}(x,y) = 2^{-1} K_{a,b,\alpha\beta}\{W_1^-(x,y) + W_2^-(x,y) + W_3^-(x,y)\},$$

where,

$$W_1^-(x,y) = -(2\alpha)\left(\frac{x}{y}\right)^{2\alpha}\frac{1}{y}\left(\frac{1}{y-x}+\frac{1}{y+x}\right) F\left(\frac{\alpha-\beta-a+b}{2},\frac{a-b+\alpha-\beta}{2},3\alpha+\beta;\ \frac{x^2}{y^2}\right),$$

$$W_2^-(x,y) = \left(\frac{x}{y}\right)^{2\alpha}\left(\frac{-1}{(y-x)^2}+\frac{-1}{(y+x)^2}\right) F\left(\frac{\alpha-\beta-a+b}{2},\frac{a-b+\alpha-\beta}{2},3\alpha+\beta;\ \frac{x^2}{y^2}\right),$$

and

$$W_3^-(x,y) = \left(\frac{x}{y}\right)^{2\alpha}\left(\frac{1}{y-x}+\frac{1}{y+x}\right)\frac{\partial}{\partial y}\left\{F\left(\frac{\alpha-\beta-a+b}{2},\frac{a-b+\alpha-\beta}{2};\ 3\alpha+\beta;\ \frac{x^2}{y^2}\right)\right\}$$

$$= -\frac{(\alpha-\beta)^2-(a-b)^2}{2(3\alpha+\beta)}\left(\frac{x}{y}\right)^{6\alpha+4\beta}\frac{1}{y}\left(\frac{1}{y-x}+\frac{1}{y+x}\right)$$

$$\times\ F\left(\frac{\alpha-\beta-a+b+2}{2},\frac{a-b+\alpha-\beta+2}{2};5\alpha+3\beta;\frac{x^2}{y^2}\right)$$

$$= -c_{a,b,\alpha,\beta}\left(\frac{x}{y}\right)^{6\alpha+4\beta}\left(\frac{1}{y-x}+\frac{1}{y+x}\right)^2 F_3 - c'_{a,b,\alpha,\beta}\left(\frac{x}{y}\right)^{8\alpha+6\beta}\left(\frac{1}{(y-x)^2}-\frac{1}{(y+x)^2}\right)F_4,$$

where

$$F_3 = F\left(\frac{\alpha-\beta-a+b}{2},\frac{a-b+\alpha-\beta+2}{2};5\alpha+3\beta;\frac{x^2}{y^2}\right),$$

$$F_4 = F\left(\frac{\alpha-\beta-a+b+2}{2},\frac{a-b+\alpha-\beta+2}{2};7\alpha+5\beta;\frac{x^2}{y^2}\right).$$

Since $0 < h < c$, it follows that $\xi \geq c - h/2 \geq c/2 \geq |x-c|/2$, which implies $\frac{1}{(x+\xi)} \leq \frac{1}{\xi} \leq \frac{2}{|x-c|}$. This inequality and $\frac{1}{|x-\xi|} \leq \frac{2}{|x-c|}$ allows us to follow the line of the proof of Case I if $(\alpha-\beta) \geq -\frac{1}{2}$, and get the inequality (3.4) in Case II. We complete the proof of Lemma 3.3.

Proof of Lemma 3.4: Let a' be an atom with $Supp\ a' \subset [0,\infty)$. It follows from Lemma 3.3 and the inequality

$$\left\|T_{\alpha,\beta}^{a,b}a'\right\|_{L^1(0,\infty)} \leq 2^{3/2}\, N\left(T_{\alpha,\beta}^{a,b}a'\right).$$

that $T_{\alpha,\beta}^{a,b}a'$ is integrable for $(\alpha-\beta) \geq -\frac{1}{2}$, $a-b \geq \frac{1}{2}$ *or* $\alpha-\beta \geq -\frac{1}{2}$, $a-b = -\frac{1}{2}$. Thus, for these $\alpha-\beta$, $a-b$, we have

$$\int_0^\infty T_{\alpha,\beta}^{a,b}\, a'(x)\, dx = \lim_{\epsilon\to 0+}\int_0^\infty e^{-\epsilon x^2}\, T_{\alpha,\beta}^{a,b}\, a'(x)\, dx.$$

By the fact

$$T_{\alpha,\beta}^{a,b}\, a'(x) = \lim_{M\to\infty}\int_0^M \mathcal{H}_{a,b}\ a'(y)(xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\, dy\ \ in\ L^2(0,\infty),$$

we have

$$\int_0^\infty T_{\alpha,\beta}^{a,b}\, a'(x)\, dx = \lim_{\epsilon\to 0+}\lim_{M\to\infty}\int_0^\infty e^{-\epsilon x^2}\int_0^m \mathcal{H}_{a,b}\, a'(y)\,(xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\, dy\, dx$$

$$= \lim_{\epsilon\to 0+}\ \lim_{M\to\infty}\int_0^\infty e^{-\epsilon x^2}\int_0^\infty\int_0^\infty a'(t)\,(yt)^{a-b} J_{a-b}(yt)\, dt\,(xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\, dy\, dx.$$

Since

$$\left|z^{p+q} J_{p'}(z)\right| \leq C, \qquad z > 0\ for\ p' \geq -\frac{1}{2}\ and\ e^{-\epsilon x^2}\, a'(t)$$

is integrable in (x,y,t) on $(0,\infty)\times(0,M)\times(0,\infty)$, it follows that

$$\int_0^\infty T_{\alpha,\beta}^{a,b}\, a'(x)\, dx = \lim_{\epsilon\to 0+}\lim_{M\to\infty}\int_0^\infty a'(t)\, B_M^{(\epsilon)}(t)\, dt$$

for $(\alpha-\beta), (a-b) \geq -\frac{1}{2}$, where

$$B_M^{(\epsilon)}(t) = \int_0^M D_t^{(\epsilon)}(y)\,dy\ ,$$

$$D_t^{(\epsilon)}(y) = \int_0^\infty e^{-\epsilon x^2}(xy)^{\alpha+\beta} J_{\alpha-\beta}\,(xy)\,dx\ \ (ty)^{a+b}\,J_{a-b}(ty).$$

To prove

$$\int_0^\infty T_{\alpha,\beta}^{a,b}\ a'(x)\,dx = 0\,,$$

we shall show the following:

(I) Let $t > 0,\ 0 < \epsilon < 1$ and $1 < M$. If $\alpha - \beta > -\frac{3}{2}$, $\neq -1$ and $a - b > -\frac{1}{2}$, $then \left|B_M^{(\epsilon)}(t)\right| \leq C$, where C depends only on $\alpha - \beta\ and\ a - b$.

(II) For every

$$t > 0, \qquad \lim_{\epsilon\to 0+}\lim_{M\to\infty} B_M^{(\epsilon)}(t) = C_{\alpha,\beta,a,b}\ ,$$

where

$$C_{\alpha,\beta,a,b} = \frac{\Gamma\left(\frac{3\alpha - 3\beta + 8}{6}\right)\Gamma(a)}{\Gamma(\alpha)\,\Gamma(2a + b)}$$

where $(\alpha - \beta) > -\frac{3}{2}, \neq -1\ and\ (a - b) > -\frac{1}{2}$.

If we show **(I)** and **(II),** then by the Lebesgue dominated convergence theorem we shall get

$$\int_0^\infty T_{\alpha,\beta}^{a,b}\ a'(x)\,dx = \int_0^\infty a'(t)\ \lim_{\epsilon\to 0+}\lim_{M\to\infty} B_M^{(\epsilon)}(t)\,dt = C_{\alpha,\beta,a,b}\int_0^\infty a'(t)\,dt = 0$$

for $(\alpha - \beta) \geq -\frac{1}{2}$ and $(\alpha - \beta) \geq \frac{1}{2}$ and the proof of Lemma 3.4 will be completed.

Let us prove **(I)** and **(II).** We shall use the formula (cf. [15,13.3(3), p.394])

$$\int_0^\infty e^{-\epsilon x^2}\,(xy)^{\alpha+\beta}\,J_{\alpha-\beta}(xy)\,dx = \frac{y^{2\alpha}\Gamma\big((4\alpha+2\beta)/2\big)}{2^{3\alpha+\beta}\ \epsilon^{((4\alpha+2\beta)/2)}\Gamma(3\alpha+\beta)}\ e^{-y^2/(4\epsilon)}$$
$$\times\ \Phi\big((3\alpha+\beta)/2\,;3\alpha+\beta\,;\ y^2/(4\epsilon)\big),$$

where $\alpha - \beta > -3/2$ and $\Phi\,(p';r';z)$ is Kummer's confluent hyper-geometric series defined by $\Phi\,(p';r';z) = \sum_{k=0}^{\infty}[(p')_k/(r')_k]\,[z^k/k!]$, for $z, p', r' \in C$, $r' \neq 0, -1, -2, \dots\dots\dots$

Since $\Phi\,(p';r';z)$ is an entire function of z, it follows that for $0 < y \leq 2\sqrt{\epsilon}$,

$$\left|\int_0^\infty e^{-\epsilon x^2}\ (xy)^{\alpha+\beta}\,J_{\alpha-\beta}(xy)\,dx\right| \leq C\,\epsilon^{-2\alpha-\beta}\,y^{2\alpha}\ , \tag{3.6}$$

when $(\alpha - \beta) > -\frac{3}{2}$. The asymptotic formula $(cf.\,[3, 6.13.1(3), Vol.\ 1,\ p.\,278])$

$\Phi\,(p';\ r';z) = \frac{\Gamma(r')}{\Gamma(p')}\,e^z\,z^{p'-r'}\,[1 + O(|z|^{-1})],\ \mathcal{R}z \to \infty\ ,\ r' \neq 0, -1, \dots .2, \dots ..$, gives, for $2\sqrt{\epsilon} \leq y$,

$$\int_0^\infty e^{-\epsilon x^2}\,(xy)^{\alpha+\beta} J_{\alpha-\beta}\,(xy)\,dx = C_{\alpha,\beta}\,y^{-1} + \mathcal{R}_\epsilon\,(y)\,,\ \ |R_\epsilon(y)| \leq C\,\epsilon\,y^{-3}, \tag{3.7}$$

if $(\alpha - \beta) > -\frac{3}{2}$, $\neq -1$, where C depends only on $\alpha - \beta$ and

$$C_{\alpha,\beta} = \frac{2^{\frac{1}{2}}\,\Gamma\left((4\alpha+2\beta)/2\right)}{\Gamma\big((2\alpha)/2\big)}.$$

Let $t > 0,\ 0 < \epsilon < 1\ \ and\ \ 1 < M$. We divide the integral

$$B_M^{(\epsilon)}(t) = \int_0^M D_t^{(\epsilon)}\,(y)\,dy$$

into two parts:

$$B_M^{(\epsilon)}(t) = \left\{\int_0^{2\sqrt{\epsilon}} + \int_{2\sqrt{\epsilon}}^{M}\right\} D_t^{(\epsilon)}\,(y)\,dy\ .$$

We begin with estimating the integral

$$\int_0^{2\sqrt{\epsilon}} D_t^{(\epsilon)}(y)\,dy\,.$$

By (3.6) and $|z^{a+b} J_{a-b}(z)| \le C$, $z > 0$ for $(a-b) \ge -\frac{1}{2}$, we have

$$\left|\int_0^{2\sqrt{\epsilon}} D_t^{(\epsilon)}(y)dy\right| \le \int_0^{2\sqrt{\epsilon}} \left|D_t^{(\epsilon)}(y)\right| dy \le C\,\epsilon^{-2\alpha-\beta} \int_0^{2\sqrt{\epsilon}} y^{2\alpha}\,dy = C \qquad (3.8)$$

for $(\alpha-\beta) > -\frac{3}{2}$ and $(\alpha-\beta) > -\frac{1}{2}$, where C depends only on $(\alpha-\beta)$ and $(a-b)$. Let $t > 0$ be fixed and let $\epsilon > 0$ be sufficiently small so that $2\sqrt{\epsilon} < \frac{1}{t}$. By (3.6) and the fact $J_{a-b}(z) = O(z^{a-b})$ $(z \to 0)$, for $a-b \ne -1, -z, \dots\dots$, $we\ have$

$$\left|\int_0^{2\sqrt{\epsilon}} D_t^{(\epsilon)}(y)\,dy\right| \le C\,\epsilon^{-2\alpha-\beta} \int_0^{2\sqrt{\epsilon}} y^{2\alpha}\,(ty)^{2a}\,dy = C\,t^{2a}\,\epsilon^{a}\,.$$

Thus for every $t > 0$, we have

$$\lim_{\epsilon\to 0} \int_0^{2\sqrt{\epsilon}} D_t^{(\epsilon)}(y)\,dy = 0 \qquad (3.9)$$

when $(\alpha-\beta) > -\frac{3}{2}$ and $(a-b) > -\frac{1}{2}$.

We next estimate the integral

$$\int_{2\sqrt{\epsilon}}^{M} D_t^{(\epsilon)}(y)\,dy.$$

By (3.7), we have

$$\int_{2\sqrt{\epsilon}}^{M} D_t^{(\epsilon)}(y)\,dy = C_{\alpha,\beta}\,U_1 + U_2 \quad ,$$

where

$$U_1 = \int_{2\sqrt{\epsilon}}^{M} (ty)^{a+b} J_{a-b}(ty)\,y^{-1}\,dy, \quad U_2 = \int_{2\sqrt{\epsilon}}^{M} (ty)^{a+b}\,J_{a-b}(ty)\,R_\epsilon(y)\,dy$$

for $(a-b) > -\frac{3}{2}$, $\ne -1$.

The integral U_2 is estimated by (3.7) and $|z^{a+b} J_{a-b}(z)| \le C$, $z > 0$ for $z > 0$ for $(a-b) \ge -\frac{1}{2}$. We have

$$|U_2| \le C\,\epsilon \int_{2\sqrt{\epsilon}}^{\infty} y^{-3}\,dy \le C \qquad (3.10)$$

for $(\alpha-\beta) > -\frac{3}{2}$, $\ne -1$ and $(a-b) \ge -\frac{1}{2}$. Let $t > 0$ be fixed, and let $\epsilon > 0$ be sufficiently small and M be sufficiently large so that $2\sqrt{\epsilon} < \frac{1}{t} < M$. We divide the integral as follow :

$$U_2 = \left\{ \int_{2\sqrt{\epsilon}}^{1/t} + \int_{1/t}^{M} \right\} (ty)^{a+b}\,J_{a+b}(ty)\,R_\epsilon(y)\,dy = U_2^1 + U_2^2\,,\ \ (say\,).$$

By the fact $J_{a-b}(z) = O(z^{a-b})$ $(z \to 0)$ $for\ a-b \ne -1, -2, \dots\dots$,
we have

$$|U_2'| \le C \int_{2\sqrt{\epsilon}}^{1/t} (ty)^{2a}\,\epsilon\,y^{-3}\,dy$$

$$\le \begin{cases} C\displaystyle\int_0^{1/t} (ty)^{2a}\,\epsilon\,y^{-3}\,dy \le C\,t^2\,\epsilon\ , (a-b > 3/2), \\ C\displaystyle\int_{2\sqrt{\epsilon}}^{1/t} t^2\,\epsilon\,y^{-1}dy \le C\,t^2\,\epsilon\left(|log\,t| + log({}^{1}\!/_{\epsilon})\right)\ (a-b = 3/2) \\ C\displaystyle\int_{2\sqrt{\epsilon}}^{\infty} (ty)^{2a}\,\epsilon\,y^{-3}\,dy \le C\,t^{2a}\,\epsilon^{a} \quad \left((a-b) < {}^{3}\!/_{2}\right) \end{cases}$$

for $(\alpha-b) > -\frac{3}{2}$, $\ne -1$ and $(a-b) \ne -1, -2 \dots\dots$ It follows from the fact that $z^{p+q} J_{\alpha-\beta}(z) = O(1)$ $(z \to \infty)$ that

$$|U_2^2| \leq C \int_{1/t}^{\infty} \epsilon\, y^{-3}\, dy \leq C\, t^2\, \epsilon .$$

Therefore, we have

$$\lim_{\epsilon\to 0} \lim_{M\to\infty} U_2 = 0 \tag{3.11}$$

for $(\alpha-\beta) > -\frac{3}{2}$, $\neq -1$ *and* $(\alpha-\beta) > -\frac{1}{2}$.

We turn to estimating U_1. We first deal with the case $2\sqrt{\epsilon} \leq \frac{1}{t} \leq M$ and divide the integral:

$$U_1 = \left\{ \int_{2\sqrt{\epsilon}}^{1/t} + \int_{1/t}^{M} (ty)^{a+b} J_{a-b}(ty)\, y^{-1}\, dy = U_1^1 + U_1^2 \ (say) . \right)$$

By the fact $J_{a-b}(z) = 0\,(z^{a-b})\ (z\to 0)$ for $a-b \neq -1, -2, \ldots\ldots,$ we have

$$|U_1^1| \leq C \int_0^{1/t} (ty)^{2a}\, y^{-1}\, dy.$$

Thus if $(a-b) > -\frac{1}{2}$, then $|U_1^1| \leq C$. Let us evaluate U_1^2 . The function satisfies

$$\frac{d}{dz} z^{-\lambda} J_\lambda(z) = -z^{-\lambda} J_{\lambda+1}(z) .$$

This and integration by parts leads to

$$U_1^2 = (-t^{-2a-4b}) \int_{1/t}^{M} y^{-2a-4b} \frac{d}{dy} \left((ty)^{a+3b} J_{-a-3b}(ty)\right) dy$$

$$= (-t^{-2a-4b}) \left[y^{-2a-4b} (ty)^{a+3b} J_{-a-3b}(ty)\right]_{1/t}^{M}$$

$$-(-t^{-2a-4b}) \int_{1/t}^{M} (ty)^{a+3b} J_{-a-3b}(ty) \frac{d}{dy} (y^{-2a-4b})\, dy$$

$$= U_1^{2,1} + U_1^{2,2} \text{ (say) .}$$

The first term $U_1^{2,1} = -(tM)^{-(a+b)} J_{-a-3b}(tM) + J_{-a-3b}(1)$ satisfies $\left|U_1^{2,1}\right| \leq C$ since $\sqrt{z} J_\lambda(z) = 0\,(1)\ (z\to\infty)$ and $1 \leq tM$. The second term

$$U_1^{2,2} = (-2a-4b) \frac{1}{t} \int_{1/t}^{M} y^2\, (ty)^{a+b} J_{-a-3b}(ty)\, dy$$

is evaluated as follows:

$$\left|U_1^{2,2}\right| \leq C\, t^{-1} \int_{1/t}^{\infty} y^{-2}\, dy \leq C.$$

Thus, we have $|U_1^2| \leq C$ and then $|U_1| \leq C$ in the case $2\sqrt{\epsilon} \leq \frac{1}{t} \leq M$. In the case $\frac{1}{t} < 2\sqrt{\epsilon}$, we have $|U_1| \leq C$ in the same way as in the estimation of U_1^2, and in the case $M \leq \frac{1}{t}$, we also have $|U_1| \leq C$ in the same way as in the estimation of U_1^1. Therefore, these and (3.10) imply

$$\left|\int_{2\sqrt{\epsilon}}^{M} D_t^{(\epsilon)}(y)\, dy\right| \leq C \tag{3.12}$$

for $(\alpha-\beta) > -3/2$, $\neq -1$ *and* $(a-b) > -\frac{1}{2}$.

Combining (3.8) and (3.12), we have **(I)**. The statement **(II)** is proved as follows : By (3.9) and (3.11), we have

$$\lim_{\epsilon\to 0+} \lim_{M\to\infty} B_M^{(\epsilon)}(t) = C_{\alpha,\beta} \lim_{\epsilon\to 0+} \lim_{M\to\infty} U_1 = C_{\alpha,\beta} \int_0^{\infty} (ty)^{a+b} J_{a-b}(ty)\, y^{-1} \times dy$$

$$= C_{\alpha,\beta} \int_0^{\infty} J_{a-b}(u)\, u^{-\frac{1}{2}}\, du = C_{\alpha,\beta,a,b}$$

for every $t > 0$ when $(\alpha - \beta) > -3/2, \quad \neq -1$ and $(a - b) > -\frac{1}{2}$. We have used $\int_0^\infty J_{a-b}\,(u)\,u^{-\frac{1}{2}}\,du = \Gamma(a)\,/\,\big(\Gamma(2a+b)\sqrt{2}\big)$ for $(a-b) > -\frac{1}{2}$.

Remarks:

(i) For $\alpha = \frac{1}{4} + \frac{\mu}{2}, \beta = \frac{1}{4} - \frac{\mu}{2}, \; a = \frac{1}{4} + \frac{\nu}{2}, \; b = \frac{1}{4} - \frac{\nu}{2}$, all the results in this paper reduce to that of Yuichi Kanjn, A transplantation theorem for the Hankel transform on the Hardy space, Tohuko Math. J. 57(2005), 231-246.

(ii)Results of Yuichi Kanjn are particular case of ours.

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