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# CHARACTERIZATION OF HANKEL TYPE KERNELS

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**Abstract**

In this paper we consider an expression involving the Bassel type function, the Neumam function and the MacDonald function and discover various combinations of these functions which are Fourier kernels or conjugate Fourier kernels. Also a large number of integration formulae are established involving these kernels.

**2000 Mathematics subject classification**

## 1.Introduction

Many authors studied Fourier kernels and developed theory by using it. In [1] they consider the manner in which Fourier kernels may be generated as solutions of ordinary differential equations. Following [5, 8] we have

$$k(x)=x^{\alpha+\beta}\left[\sin(\alpha-\beta)\frac{\pi}{2}J_{\alpha-\beta}(x)+\cos(\alpha-\beta)\frac{\pi}{2}\left(Y_{\alpha-\beta}(x)+\frac{2}{\pi}K_{\alpha-\beta}(x)\right)\right]$$

Where $J_{\lambda}(x)$ is the Bessel type function, $Y_{\lambda}(x)$ is the Neumann type function and $K_{\lambda}(x)$ is the MacDonald type function. In this paper we follow a different line of thought. We inquire which expressions of the type

$k(x)=x^{\alpha+\beta}\left[AJ_{\alpha-\beta}(x)+BY_{\alpha-\beta}(x)+CK_{\alpha-\beta}(x)\right]$ ( $A,B,C$ Being constants) are Fourier type kernels or have conjugate kernels of the same form. In this manner we study some new Fourier type kernels and others which have simple looking conjugate.We also establish a large number of integration formulae, involving the function $k(x)$ and its conjugate. Many of these formulae are believed unavailable in the literature. Throughout, we print out various known results as special cases of our general results.

## 2. Preliminaries

We shall mention below a few known results and definitions from the theory of Mellin transforms, which will be needed later. All such results can be found in [7].

A function $F(s), s=c+it, -\infty<t<\infty, a<c<b,$ is said to be the Mellin transform of $f(x)$ if

$$F(s)=\int_0^\infty f(x)x^{s-1}dx$$
$$=\mathrm{M}\left[f(x);s\right]$$

Conversely we call

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)x^{-s}ds$$
$$=\mathrm{M}^{-1}\left[F(s);x\right]$$

the inverse Mellin transform of $F(s)$.

An important result in the theory of Mellin transform is the Panseval theorem: If $F(s)$ and $K(s)$ are the Mellin transforms of the functions $f(x)$ and $k(x)$ respectively, then under appropriate conditions

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}K(s)F(1-s)x^{-s}ds=\int_0^\infty k(xt)f(t)dt \tag{2.1}$$

A direct consequence of the Parseval theorem is that if $K(s)F(1-s)=G(s)$ (2.2)

where $K(s), F(s)$, $G(s)$ denote the Mellin transform of $k(x)$, $f(x)$ and $g(x)$ respectively, then in suitable strip of the s-plane, we have

$$\int_0^\infty k(xt)f(t)dt=g(x) \tag{2.3}$$

Next, (2.3) implies (2.2), and we call $g$ to be $k$ -transform of $f$ . If further the inversion formulae

$$\int_0^\infty h(xt)g(t)dt=f(x), \tag{2.4}$$

Involving the kernel $h(x)$, holds, then $k$ and $h$ are said to be conjugate of each other. Also their Mellin transform satisfy the functional equation

$$K(s)H(1-s)=1$$

In some strip of the s-plane. If instead of (2.4), we have the inversion formula

$$\int_0^\infty k(xt)g(t)dt=f(x) \qquad (2.5)$$

along with (2.3), then $k$ is said to be self conjugate or a Fourier kernel. Also its Mellin transform satisfies the equation

$$K(s)K(1-s)=1$$

Thus, If the equations (2.3) and (2.4) holds simultaneously, then we shall call $k(x)$ and $h(x)$, conjugate kernels. If on the other hand equations (2.3) and (2.5) hold, then $k(x)$ is said to be a self conjugate kernels. If for suitable $f$,

$$\int_0^\infty k(xt)f(t)dt=\pm f(t),$$

then $f$ is said to be an Eigen function of the operator $k$, corresponding to the eigenvalue $\pm 1$ respectively. It should be noted that if the operator $k$ is Fourier kernel, then it has only these two eigenvalues.

## 3. The kernels

We consider the function

$$k(x)=x^{\alpha+\beta}\left[AJ_{\alpha-\beta}(x)+BY_{\alpha-\beta}(x)+CK_{\alpha-\beta}(x)\right],$$

where $A,B,C$ are real constants. We may assign appropriate values to these constants so that $k(x)$ is either self conjugate or has a conjugate of the same type. Our first task will be to determine those values of $A,B,C$. The technique we shall employ to find those suitable values, consists of using results from Mellin transform theory. The crucial part of our procedure is to express the function $K(s)$, the Mellin transform of $k(x)$, as a rational expression of Gamma functions.

Now, making use of Mellin transform of the functions $x^{\alpha+\beta}J_{\alpha-\beta}(x)$, $x^{\alpha+\beta}Y_{\alpha-\beta}(x)$ and $x^{\alpha+\beta}K_{\alpha-\beta}(x)$, [3], the Mellin transform of $k(x)$ is then given by

$$K(s)=\mathrm{M}\left[k(x);s\right]$$

$$=\frac{1}{\pi}2^{s-\frac{1}{2}}\Gamma\left(\alpha+\frac{s}{2}\right)\Gamma\left(\beta+\frac{s}{2}\right)\left[A\sin\pi\left(\beta+\frac{s}{2}\right)-B\cos\pi\left(\beta+\frac{s}{2}\right)+C\right]$$

$$\text{where } |\alpha-\beta|-\frac{1}{2}<\operatorname{Re}s<1.$$

In order to consolidate the bracketed terms into a single term, an appropriate choice for the constants is that

$$A=\cos\theta\pi,\ B=\sin\theta\pi,\ C=\frac{2}{\pi}\sin a\pi,$$

where $\theta$ and $a$ are arbitrary. Then one can write after some simplification,

$$K(s)=\frac{1}{\pi}2^{s-\frac{1}{2}}\Gamma\left(\alpha+\frac{s}{2}\right)\Gamma\left(\beta+\frac{s}{2}\right)\sin\frac{\pi}{4}(2\beta-2\theta+2a+s)\sin\frac{\pi}{4}\left(2(2\alpha+\beta)+2\theta+2a-s\right)$$

Now using the functional equation $\Gamma(z)\Gamma(1-z)=\pi\operatorname{cosec}\pi z$ And the duplication formula for $\Gamma(2z)$, we obtain,

$$K(s)=\frac{2^{2s-1}\Gamma\left(\frac{\alpha}{2}+\frac{s}{4}\right)\Gamma\left(\frac{\beta}{2}+\frac{s}{4}\right)\Gamma\left(\frac{3}{2}\alpha+\beta+\frac{s}{4}\right)\Gamma\left(\alpha+\frac{3}{2}\beta+\frac{s}{4}\right)}{\Gamma\left(\alpha+\frac{3}{2}\beta-A_1+\frac{s}{4}\right)\Gamma\left(\alpha+\frac{\beta}{2}+A_1-\frac{s}{4}\right)\Gamma\left(\frac{\beta}{2}-B_1+\frac{s}{4}\right)\Gamma\left(2\alpha+\frac{3}{2}\beta+B_1-\frac{s}{4}\right)} \qquad (3.1)$$

where $|\alpha-\beta|-\frac{1}{2}<\operatorname{Re}s<1$, $A_1=\frac{1}{2}(\theta+a)$ and $B_1=\frac{1}{2}(\theta-a)$.

Also the corresponding form of $k(x)$ using the above values of $A,B,C$, is then

$$k(x)=M^{-1}\left[K(s);x\right]$$

$$=x^{\alpha+\beta}\left[\cos\theta\pi J_{\alpha-\beta}(x)+\sin\theta\pi Y_{\alpha-\beta}(x)+\frac{2}{\pi}\sin a\pi K_{\alpha-\beta}(x)\right] \tag{3.2}$$

To determine the function $h(x)$, the conjugate of $k(x)$, we consider the functional equation

$$H(s)K(1-s)=1$$

where $H(s)$ and $K(s)$ are the Mellin transform of $h(x)$ and $k(x)$, respectively; hence

$$H(s)=\frac{2^{2s-1}\Gamma\left(\frac{\alpha}{2}+A_1+\frac{s}{4}\right)\Gamma\left(\frac{3}{2}\alpha+2\beta-A_1-\frac{s}{4}\right)\Gamma\left(\frac{3}{2}\alpha+\beta+B_1+\frac{s}{4}\right)\Gamma\left(\frac{\alpha}{2}+\beta-B_1-\frac{s}{4}\right)}{\Gamma\left(\alpha+\frac{\beta}{2}-\frac{s}{4}\right)\Gamma\left(2\alpha+\frac{3}{2}\beta-\frac{s}{4}\right)\Gamma\left(\frac{\alpha}{2}+\beta-\frac{s}{4}\right)\Gamma\left(\frac{3}{2}\alpha+2\beta-\frac{s}{4}\right)} \tag{3.3}$$

Now in suitable strip of the s-plane, we have

$$h(x)=M^{-1}\left[H(s);x\right],$$

which can be shown by complex integration to be the sum of two hyper geometric series, eventually giving us the conjugate of the function $k(x)$.In the next two sections, we shall explore situations giving rise to four special cases. These cases are of particular interest since they lead to a simpler representation of the conjugate function $h(x)$. In cases $h(x)$ coincides with $k(x)$, defining a self conjugate kernel. First we shall discuss self conjugate kernels.

**4. Self-Conjugate kernels**:

Let $\theta=\frac{1}{2}(\alpha+3\beta)$ and $a=\frac{1}{2}(3\alpha+\beta)$. Then $A_1=\frac{1}{2}$, $B_1=-\frac{1}{2}(\alpha-\beta)$ and from (3.1) and (3.3), we have

$$K_1(s)=H_1(s)=\frac{2^{2s-1}\Gamma\left(\frac{3}{2}\alpha+\beta+\frac{s}{4}\right)\Gamma\left(\alpha+\frac{3}{2}\beta+\frac{s}{4}\right)}{\Gamma\left(2\alpha+\frac{3}{2}\beta-\frac{s}{4}\right)\Gamma\left(\frac{3}{2}\alpha+2\beta-\frac{s}{4}\right)} \tag{4.1}$$

where $|\alpha-\beta|-\frac{5}{2}<\operatorname{Re}s<1,$ and $K_1(s)K_1(1-s)=1.$

Then from (3.2)

$$k_1(x)=x^{\alpha+\beta}\left[\sin(\alpha-\beta)\frac{\pi}{2}J_{\alpha-\beta}(x)+\cos(\alpha-\beta)\frac{\pi}{2}\left(Y_{\alpha-\beta}(x)+\frac{2}{\pi}K_{\alpha-\beta}(x)\right)\right] \tag{4.2}$$

and it defines a self-conjugate kernel , i.e. $k_1(x)$ is a Fourier kernel [1]. An interesting special case of the above kernel occurs when $\alpha-\beta=\pm\frac{1}{2}$, when $k_1(x)$ becomes [5],

$$\frac{1}{\sqrt{\pi}}\left[\sin x-\cos x+e^{-x}\right].$$

Further we shall establish various integration formulae involving the function $k_1(x)$. These formulae are derived as a result of suitable decomposition of the Mellin transform function $K_1(s)$. For instance let us define a function $F$ , by

$$F(s)=2^{-\frac{1}{2}+s}\frac{\Gamma\left(\alpha+\frac{3}{2}\beta+\frac{s}{4}\right)}{\Gamma\left(2\alpha+\frac{3}{2}\beta-\frac{s}{4}\right)},$$

where $s=c+it,-\infty<t<\infty$ and $2\beta<c<2(3\alpha+4\beta)$. Then from (4.1), we deduce that

$$K_1(s)F(1-s)=F(s) \qquad (4.3)$$

Now since [3],

$$F(s)=M\left[\frac{1}{\sqrt{\pi}}2^{3\alpha+\beta}x^{2\beta}\sin\left(\frac{1}{2}x^2\right);s\right],$$

where $-4\alpha-6\beta<\operatorname{Re}s<4\alpha+2\beta$ then due to the result (2.3),the functional equation (4.3) implies that, on $\operatorname{Re}s=\frac{1}{2}$,

$$\frac{1}{\sqrt{\pi}}2^{3\alpha+\beta}\int_0^\infty t^{2\beta}\sin\left(\frac{1}{2}t^2\right)k_1(xt)\,dt=\frac{1}{\sqrt{\pi}}2^{3\alpha+\beta}x^{2\beta}\sin\left(\frac{1}{2}x^2\right), \qquad (4.4)$$

$|\alpha-\beta|<3$.

Hence $\frac{1}{\sqrt{\pi}}2^{3\alpha+\beta}x^{2\beta}\sin\left(\frac{1}{2}x^2\right)$ is an Eigen function of the operator $k_1(x)$, corresponding to the eigenvalue 1. Letting $(\alpha-\beta)=\frac{1}{2}$, gives the special case

$$\frac{2}{\sqrt{\pi}}\int_0^\infty\sin\left(\frac{1}{2}t^2\right)\left(\sin xt-\cos xt+e^{-xt}\right)dt=\sin\left(\frac{1}{2}x^2\right) \qquad (4.5)$$

Letting $\alpha-\beta=1$ and $2$ in (4.4), we obtain two more interesting special cases, which are

$$\int_0^\infty t^{-(\alpha+\beta)}\sin\left(\frac{1}{2}t^2\right)(xt)^{(\alpha+\beta)}J_1(xt)\,dt=x^{-(\alpha+\beta)}\sin\left(\frac{1}{2}x^2\right), \qquad (4.6)$$

(See [4, P.19 (16)]), and

$$-\int_0^\infty t^{-\frac{3}{2}}\sin\left(\frac{1}{2}t^2\right)(xt)^{(\alpha+\beta)}\left(Y_2(xt)+\frac{2}{\pi}K_2(xt)\right)dt=x^{-\frac{3}{2}}\sin\left(\frac{1}{2}x^2\right). \qquad (4.7)$$

Further if we define $F$ by

$$F(s)=2^{-\frac{1}{2}+s}\Gamma\left(\frac{3}{2}\alpha+\beta+\frac{s}{4}\right)\Gamma\left(\alpha+\frac{3}{2}\beta+\frac{s}{4}\right),$$

where $\mathrm{Re}(s)>-\frac{5}{2}+|\alpha-\beta|,$ then from (4.1), we have

$$K_1(s)F(1-s)=F(s),\ |\alpha-\beta|-\frac{5}{2}<\mathrm{Re}(s)<1 \tag{4.8}$$

And since,

$$F(s)=M\left[x^{5/2}K_{\frac{\alpha-\beta}{2}}\left(\frac{1}{2}x^2\right);s\right],$$

then (4.8), due to (2.1), implies

$$\int_0^\infty t^{5/2}K_{\frac{\alpha-\beta}{2}}\left(\frac{1}{2}t^2\right)k_1(xt)\,dt=x^{5/2}K_{\frac{\alpha-\beta}{2}}\left(\frac{1}{2}x^2\right),|\alpha-\beta|<3 \tag{4.9}$$

giving another Eigen function $x^{5/2}K_{\frac{\alpha-\beta}{2}}\left(\frac{1}{2}x^2\right)$ of the operator $k_1(x)$.

Again letting $\alpha-\beta=1$ and $2$, We obtain special cases of (4.9), which are respectively

$$\int_0^\infty t^2e^{-\frac{1}{2}t^2}J_1(xt)\,dt=xe^{-\frac{1}{2}x^2}, \quad [4, p.19(8)] \tag{4.10}$$

and

$$-\int_0^\infty t^2K_1\left(\frac{1}{2}t^2\right)\left(Y_2(xt)+\frac{2}{\pi}K_2(xt)\right)dt=x^2K_1\left(\frac{1}{2}x^2\right) \tag{4.11}$$

In general, in order that $f(x)$ should be an Eigen function of the operator $k_1$ corresponding to the eigenvalue 1, $F(s),\zeta=\sigma+i\xi$, the Mellin transform of $f$ should be of the form

$$F(s)=2^{s-\frac{1}{2}}\Gamma\left(\frac{3}{2}\alpha+\beta+\frac{s}{4}\right)\Gamma\left(\alpha+\frac{3}{2}\beta+\frac{s}{4}\right)\Psi(s),$$

where $\Psi(s)=\Psi(1-s)$. The Eigen functions mentioned above in (4.4) and (4.9) are special cases when

$$\Psi(s)=\frac{1}{\Gamma\left(2\alpha+\frac{3}{2}\beta-\frac{s}{4}\right)\Gamma\left(\frac{3}{2}\alpha+\beta+\frac{s}{4}\right)}$$

and when

$\Psi(s)=1$ respectively.

Now we define functions $F$ and $G$ by

$$F(s)=\frac{2^{s-\frac{1}{2}}\Gamma\left(\alpha+\frac{3}{2}\beta+\frac{s}{4}\right)}{\Gamma\left(\frac{3}{2}\alpha+2\beta-\frac{s}{4}\right)}$$

and

$$G(s)=2^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{3}{2}\alpha+\beta+\frac{s}{4}\right)}{\Gamma\left(2\alpha+\frac{3}{2}\beta-\frac{s}{4}\right)}$$

Then from (4.1), we have the functional equation

$$K_1(s)F(1-s)=G(s). \tag{4.12}$$

Since,

$$F(s)=M\left[x^{3/2}J_{\frac{\alpha}{2}+\frac{3}{2}\beta}\left(\frac{1}{2}x^2\right);s\right],\ -2(2\alpha+3\beta)<\operatorname{Re}s<\frac{3}{2}$$

and

$$G(s)=M\left[x^{3/2}J_{\frac{3}{2}\alpha+\frac{\beta}{2}}\left(\frac{1}{2}x^2\right);s\right],\ -2(3\alpha+2\beta)<\operatorname{Re}s<\frac{3}{2},$$

hence due to the result (2.1), (4.12) implies the equation

$$\int_0^\infty t^{3/2}J_{\frac{\alpha}{2}+\frac{3}{2}\beta}\left(\frac{1}{2}t^2\right)k_1(xt)\,dt=x^{3/2}J_{\frac{3}{2}\alpha+\frac{\beta}{2}}\left(\frac{1}{2}x^2\right),\ \ |\alpha-\beta|<3. \tag{4.13}$$

Next, $k_1$ is self-conjugate, therefore the inversion formula gives

$$\int_0^\infty t^{3/2}J_{\frac{3}{2}\alpha+\frac{\beta}{2}}\left(\frac{1}{2}t^2\right)k_1(xt)\,dt=x^{3/2}J_{\frac{\alpha}{2}+\frac{3}{2}\beta}\left(\frac{1}{2}x^2\right). \tag{4.14}$$

This establishes the pair $x^{3/2}J_{\frac{\alpha}{2}+\frac{3}{2}\beta}\left(\frac{1}{2}x^2\right)$, $x^{3/2}J_{\frac{3}{2}\alpha+\frac{\beta}{2}}\left(\frac{1}{2}x^2\right)$ as $k_1$-transforms of each other.

Some special cases of (4.13), when $\alpha-\beta=0,1$ and $2$, are respectively

$$\int_0^\infty t\sin\left(\frac{1}{2}t^2\right)\left(Y_0(xt)+\frac{2}{\pi}K_0(xt)\right)dt=\sin\left(\frac{1}{2}x^2\right) \tag{4.15}$$

$$\int_0^\infty t^2 J_0\left(\frac{1}{2}t^2\right) J_1(xt)\,dt = xJ_1\left(\frac{1}{2}x^2\right), \qquad [6, p.215(3)], \tag{4.16}$$

and

$$\int_0^\infty t\cos\left(\frac{1}{2}t^2\right)\left(Y_2(xt)+\frac{2}{\pi}K_2(xt)\right)dt = \cos\left(\frac{1}{2}x^2\right)-\left(\frac{1}{2}x^2\right)\sin\left(\frac{1}{2}x^2\right) \tag{4.17}$$

Next, if we put

$\theta = -\frac{1}{2}(3\alpha+\beta)$ and $a = \frac{1}{2}(3\alpha+\beta)$ in (3.1) and (3.3), then

$$K_2(s) = H_2(s) = 2^{2s-1}\frac{\Gamma\left(\frac{\alpha}{2}+\frac{s}{4}\right)\Gamma\left(\frac{\beta}{2}+\frac{s}{4}\right)}{\Gamma\left(\alpha+\frac{\beta}{2}-\frac{s}{4}\right)\Gamma\left(\frac{\alpha}{2}+\beta-\frac{s}{4}\right)}$$

Satisfying the equation

$$K_2(s)K_2(1-s) = 1,\ |\alpha-\beta|-\frac{1}{2} < \operatorname{Re} s < 1.$$

Also from (3.2), we have

$$k_2(x) = M^{-1}\left[K_2(s);x\right] = -x^{\alpha+\beta}\left[\sin(\alpha-\beta)\frac{\pi}{2}J_{\alpha-\beta}(x)+\cos(\alpha-\beta)\frac{\pi}{2}\left(Y_{\alpha-\beta}(x)-\frac{2}{\pi}K_{\alpha-\beta}(x)\right)\right], \tag{4.18}$$

and it defines a self-conjugate kernel.

Note that if $\alpha-\beta = \frac{1}{2}$, then we obtain

$$k_2(x) = \frac{1}{\sqrt{\pi}}\left(\cos x - \sin x + e^{-x}\right),$$

An intersecting special case [5].

Various integration formulae, involving the function $k_2(x)$, are given below, again as a result of different decompositions of the function $K_2(s)$. First we define $F$ by

$$F(s) = 2^{-\frac{1}{2}+s}\frac{\Gamma\left(\frac{\beta}{2}+\frac{s}{4}\right)}{\Gamma\left(\alpha+\frac{\beta}{2}-\frac{s}{4}\right)}.$$

Then

$$K_2(s)F(1-s) = F(s). \tag{4.19}$$

Now,

$$F(s) = M\left[\frac{2}{\sqrt{\pi}} x^{2\beta} \cos\left(\frac{1}{2}x^2\right); s\right], \ -2\beta < \operatorname{Re} s < 2(2\alpha + \beta).$$

The due to (2.1), the functional equation (4.19), on $\operatorname{Re} s = \frac{1}{2}$, implies that

$$\int_0^\infty t^{2\beta} \cos\left(\frac{1}{2}t^2\right) k_2(xt)\, dt = x^{2\beta} \cos\left(\frac{1}{2}x^2\right), \ |\alpha - \beta| < 1 \tag{4.20}$$

i.e. $x^{2\beta} \cos\left(\frac{1}{2}x^2\right)$ is an Eigen function of the operator $k_2(x)$, defined by (4.18).A special case can be derived from (4.20) by setting $|\alpha - \beta| = \frac{1}{2}$,

$$\frac{1}{\sqrt{\pi}} \int_0^\infty \cos\left(\frac{1}{2}t^2\right)\left(\cos xt - \sin xt + e^{-xt}\right) dt = \cos\left(\frac{1}{2}x^2\right) \tag{4.21}$$

It is intersecting to compare this result with (4.5).Another Eigen function of the kernel $k_2(x)$, can be obtained by letting

$$F(s) = 2^{-\frac{1}{2}+s}\,\Gamma\left(\frac{\alpha}{2} + \frac{s}{4}\right)\Gamma\left(\frac{\beta}{2} + \frac{s}{4}\right)$$

then,

$$K_2(s) F(1-s) = F(s)$$

where

$$F(s) = M\left[x^{1/2} K_{\frac{\alpha-\beta}{2}}\left(\frac{1}{2}x^2\right); s\right], \ \operatorname{Re} s > |\alpha - \beta| - \frac{1}{2},$$

implies

$$\int_0^\infty t^{1/2} K_{\frac{\alpha-\beta}{2}}\left(\frac{1}{2}t^2\right) k_2(xt)\, dt = x^{\alpha+\beta} K_{\frac{\alpha-\beta}{2}}\left(\frac{1}{2}x^2\right), \ |\alpha - \beta| < 1. \tag{4.22}$$

Letting $\alpha - \beta = 0$ and $\frac{1}{2}$, (4.22) reduces to, respectively,

$$\int_0^\infty t K_0\left(\frac{1}{2}t^2\right)\left(Y_0(xt) - \frac{2}{\pi} K_0(xt)\right) dt = -K_0\left(\frac{1}{2}x^2\right) \tag{4.23}$$

and

$$\frac{1}{\sqrt{\pi}}\int_0^\infty t^{\alpha+\beta}K_{1/4}\left(\frac{1}{2}t^2\right)\left(\cos xt-\sin xt-e^{xt}\right)dt=x^{\alpha+\beta}K_{1/4}\left(\frac{1}{2}x^2\right) \tag{4.24}$$

As before, the Eigen functions $f$ of the operator $k_2$, can be characterized, by expressing its Mellin transform as

$$F(s)=\Gamma\left(\frac{\alpha}{2}+\frac{s}{4}\right)\Gamma\left(\frac{\beta}{2}+\frac{s}{4}\right)\Psi(s),$$

where $\Psi(s)=\Psi(1-s)$. Letting

$$\Psi(s)=1$$

and

$$\Psi(s)=\frac{1}{\Gamma\left(\alpha+\frac{\beta}{2}-\frac{s}{4}\right)\Gamma\left(\frac{\alpha}{2}+\beta-\frac{s}{4}\right)},$$

give us the Eigen functions mentioned above in (4.20) and (4.22) respectively. Finally, we define $F$ and $G$ by

$$F(s)=2^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{\beta}{2}+\frac{s}{4}\right)}{\Gamma\left(\frac{\alpha}{2}+\beta-\frac{s}{4}\right)}$$

and

$$G(s)=2^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{\alpha}{2}+\frac{s}{4}\right)}{\Gamma\left(\alpha+\frac{\beta}{2}-\frac{s}{4}\right)}$$

then

$$K_2(s)F(1-s)=G(s) \tag{4.25}$$

where

$$F(s)=M\left[x^{3/2}J_{-\frac{3}{2}\alpha-\frac{\beta}{2}}\left(\frac{1}{2}x^2\right);s\right],\ -2\beta<\operatorname{Re}s<\frac{3}{2}$$

and

$$G(s)=M\left[x^{3/2}J_{-\frac{\alpha}{2}-\frac{3}{2}\beta}\left(\frac{1}{2}x^2\right);s\right],\ -2\alpha<\operatorname{Re}s<\frac{3}{2}.$$

Hence from (4.25), we obtain, on $\operatorname{Re}s=\frac{1}{2}$,

$$\int_0^\infty t^{3/2} J_{-\frac{3}{2}\alpha-\frac{\beta}{2}}\left(\frac{1}{2}t^2\right)k_2(xt)\,dt = x^{3/2} J_{-\frac{\alpha}{2}-\frac{3}{2}\beta}\left(\frac{1}{2}x^2\right),\ (\alpha-\beta)<1 \tag{4.26}$$

and conversely,

$$\int_0^\infty t^{3/2} J_{-\frac{\alpha}{2}-\frac{3}{2}\beta}\left(\frac{1}{2}t^2\right)k_2(xt)\,dt = x^{3/2} J_{-\frac{3}{2}\alpha-\frac{\beta}{2}}\left(\frac{1}{2}x^2\right),\ (\alpha-\beta)>-1 \tag{4.27}$$

Putting $\alpha-\beta=0$ and $\frac{1}{2}$ in (4.26), we obtain respectively,

$$\int_0^\infty t^{3/2} J_{-2\alpha}\left(\frac{1}{2}t^2\right)k_2(xt)\,dt = x^{3/2} J_{-2\alpha}\left(\frac{1}{2}x^2\right) \tag{4.28}$$

and

$$\int_0^\infty t^{3/2} J_{-\frac{5}{2}\alpha+\frac{1}{2}}\left(\frac{1}{2}t^2\right)k_2(xt)\,dt = x^{3/2} J_{-2\alpha+\frac{3}{4}}\left(\frac{1}{2}x^2\right) \tag{4.29}$$

**5. Conjugate kernels**:

If we put $\theta=-\frac{1}{2}(\alpha-\beta)$ and $a=1+\frac{1}{2}(\alpha-\beta)$ in the equations (3.1) and (3.3), we obtain

$$K_3(s)=\frac{2^{2s-1}\Gamma\left(\frac{\alpha}{2}+\frac{s}{4}\right)\Gamma\left(\frac{3}{2}\alpha+\beta+\frac{s}{4}\right)\Gamma\left(\alpha+\frac{3}{2}\beta+\frac{s}{4}\right)}{\Gamma\left(2\alpha+\frac{3}{2}\beta-\frac{s}{4}\right)\Gamma\left(\frac{3}{4}\alpha+\frac{\beta}{4}+\frac{s}{4}\right)\Gamma\left(\frac{5}{4}\alpha+\frac{7}{4}\beta-\frac{s}{4}\right)}$$

and

$$H_3(s)=\frac{2^{2s-1}\Gamma\left(\frac{3}{2}\alpha+\beta+\frac{s}{4}\right)\Gamma\left(\frac{3}{4}\alpha+\frac{5}{4}\beta+\frac{s}{4}\right)\Gamma\left(\frac{5}{4}\alpha+\frac{3}{4}\beta-\frac{s}{4}\right)}{\Gamma\left(\alpha+\frac{\beta}{2}-\frac{s}{4}\right)\Gamma\left(2\alpha+\frac{3}{2}\beta-\frac{s}{4}\right)\Gamma\left(\frac{3}{2}\alpha+2\beta-\frac{s}{4}\right)}$$

So that

$$H_3(s)K_3(1-s)=1.$$

Then from (3.2), we have

$$k_3(x)=x^{\alpha+\beta}\left[\cos\frac{1}{2}(\alpha-\beta)\pi J_{\alpha-\beta}(x)-\sin\frac{1}{2}(\alpha-\beta)\pi\left(Y_{\alpha-\beta}(x)+\frac{2}{\pi}K_{\alpha-\beta}(x)\right)\right], \tag{5.1}$$

where $|\alpha-\beta|-\frac{1}{2}<\operatorname{Re}s<1$. It is now a simple matter to evaluate $h_3(x)$, which is the conjugate of $k_3(x)$ and is given by

$$h_3(x) = M^{-1}\left[H_3(s):x\right]$$

$$= x^{\alpha+\beta}\left[\cos\frac{1}{2}(\alpha-\beta)\pi J_{\alpha-\beta}(x) - \sin\frac{1}{2}(\alpha-\beta)\pi\left(Y_{\alpha-\beta}(x) - \frac{2}{\pi}K_{\alpha-\beta}(x)\right)\right], \qquad (5.2)$$

for $|\alpha-\beta| - \frac{1}{2} < \operatorname{Re} s < 1$. Thus we have established a pair of conjugate kernels $k_3(x)$ and $h_3(x)$. As a special case when $\alpha-\beta = \pm\frac{1}{2}$, we have pairs of conjugate kernels,

$$\frac{1}{\sqrt{\pi}}\left(\cos t + \sin t \mp e^{-x}\right). \qquad (5.3)$$

By employing the technique of the previous sections, we arrive at the following integration formulae involving the kernel $k_3(x)$.

Integrals involving $h_3(x)$ can easily be obtained by the inversion formulae (2.4). We believe that these are all new results.

$$\int_0^\infty t^{\alpha+\beta} J_{\frac{-(\alpha-\beta)}{2}}\left(\frac{1}{2}t^2\right) k_3(xt)\,dt = x^{\alpha+\beta} J_{\frac{\alpha-\beta}{2}}\left(\frac{1}{2}x^2\right),\ (\alpha-\beta) < 2 \qquad (5.4)$$

$$\int_0^\infty t^{\alpha+\beta} J_{\frac{\alpha-\beta}{2}}\left(\frac{1}{2}t^2\right) h_3(xt)\,dt = x^{\alpha+\beta} J_{\frac{-(\alpha-\beta)}{2}}\left(\frac{1}{2}x^2\right),\ (\alpha-\beta) \ge -2 \qquad (5.5)$$

If $\alpha-\beta = 2n,\ n = 0,1,2,\cdots$, then [4, p.56(1)], the equation (5.5) gives,

$$\int_0^\infty t J_n\left(\frac{1}{2}t^2\right) J_{2n}(xt)\,dt = J_n\left(\frac{1}{2}x^2\right). \qquad (5.6)$$

If $\alpha-\beta = 1$, then from (5.4) and (5.5), we have respectively,

$$-\int_0^\infty \cos\left(\frac{1}{2}t^2\right)\left(Y_1(xt) + \frac{2}{\pi}K_1(xt)\right)dt = \frac{1}{x}\sin\left(\frac{1}{2}x^2\right), \qquad (5.7)$$

and

$$-\int_0^\infty \sin\left(\frac{1}{2}t^2\right)\left(Y_1(xt) - \frac{2}{\pi}K_1(xt)\right)dt = \frac{1}{x}\cos\left(\frac{1}{2}x^2\right) \qquad (5.8)$$

Also,

$$\int_0^\infty t^{2\beta}\cos\left(\frac{1}{2}t^2\right) k_3(xt)\,dt = x^{2\beta}\sin\left(\frac{1}{2}x^2\right),\ -\frac{1}{2} < (\alpha-\beta) < 2 \qquad (5.9)$$

If $\alpha-\beta = 0$, (5.9) gives, [4, p.38(40)],

$$\int_0^\infty t\cos\left(\frac{1}{2}t^2\right)J_0(xt)\,dt = \sin\left(\frac{1}{2}x^2\right) \tag{5.10}$$

It is also easy to establish that

$$\int_0^\infty t^{3(\alpha+\beta)}K_{\frac{3}{2}\alpha+\frac{\beta}{2}}\left(\frac{1}{2}x^2\right)dx,\ |\alpha-\beta|<2, \tag{5.11}$$

Letting $\alpha-\beta=0$, we have, [5, p.29 (10)],

$$\int_0^\infty te^{-\left(\frac{1}{2}t^2\right)}J_0(xt)\,dt = e^{-\left(\frac{1}{2}x^2\right)} \tag{5.12}$$

Our last pair of conjugate kernels is obtained if we set $a=0$ in (3.1) and (3.3). Then for $|\alpha-\beta|-\frac{1}{2}<\operatorname{Re}s<1$,

$$K_4(s)=2^{s-\frac{1}{2}}\frac{\Gamma\left(\alpha+\frac{s}{2}\right)\Gamma\left(\beta+\frac{s}{2}\right)}{\Gamma\left(2\alpha+\beta+\theta-\frac{s}{2}\right)\Gamma\left(\beta-\theta+\frac{s}{2}\right)},$$

and for $0<\operatorname{Re}s<1$, $|\alpha-\beta+2\theta|<\frac{3}{2}$,

$$H_4(s)=2^{s-\frac{1}{2}}\frac{\Gamma\left(\alpha+\theta+\frac{s}{2}\right)\Gamma\left(\alpha+2\beta-\theta-\frac{s}{2}\right)}{\Gamma\left(2\alpha+\beta-\frac{s}{4}\right)\Gamma\left(\alpha+2\beta-\frac{s}{2}\right)},$$

So that

$$H_4(s)K_4(1-s)=1$$

Thus in an appropriate strip of the s-plane, from (3.2),

$$k_4(x)=M^{-1}\left[K_4(s):x\right]$$

$$=x^{\alpha+\beta}\left[\cos\theta\pi J_{\alpha-\beta}(x)-\sin\theta\pi Y_{\alpha-\beta}(x)\right] \tag{5.13}$$

Also, using complex integration, we can find that [3, p.353 (43)],

$$h_4(x)=M^{-1}\left[H_4(s):x\right]$$

$$=\sqrt{2}G_{13}^{11}\left[\frac{1}{4}x^2\,\Big|_{b_1,b_2,b_3}^{a_1}\right], \tag{5.14}$$

where $a_1=b_1=\alpha+\theta,\ b_2=\beta,\ b_3=\alpha,$ and $G$ being the Meijer's $G$-function.

Note that alternatively, [3, p.379],

$$h_4(x) = \frac{x^{\alpha+\beta}}{\Gamma(1+\theta)\Gamma(3\alpha+\beta+\theta)}\left[\frac{1}{2}x\right]^{\alpha-\beta+2\theta} {}_1F_2\left(1;1+\theta,3\alpha+\beta+\theta,-\frac{1}{4}x^2\right) \tag{5.15}$$

It is now simple to see that if $\theta = \frac{1}{2}$, then we obtain

$$k_4(x) = x^{\alpha+\beta} Y_{\alpha-\beta}(x) \tag{5.16}$$

and [3, p.380]

$$h_4(x) = x^{\alpha+\beta} H_{\alpha-\beta}(x) \tag{5.17}$$

a pair of well-known conjugate kernels, $H_{\alpha-\beta}$ being Struve's function [7, p.215(2)].

Finally we shall list a few integration formulae involving the operator $k_4(x)$. Integrals involving $h_4(x)$ can be written by the usual inversion formulae of the type (2.4).

$$\int_0^\infty t^{\alpha+\beta+\theta} J_{\alpha-\beta+\theta}(t) k_4(xt)\,dt = \frac{2^{1+\theta}}{\Gamma(-\theta)}(1-x^2)^{-1-\theta} x^{2\beta} H(1-x),\ 3\alpha+\beta+\theta > 0, \tag{5.18}$$

where $-1 \le \theta < 0$ and $H$ is the Heaviside function.

If $\theta = -\frac{1}{2}$, then [6, p.272(4)],

$$\int_0^\infty (xt)^{\alpha+\beta} J_{-2\beta}(t) Y_{\alpha-\beta}(xt)\,dt = -\sqrt{\frac{2}{\pi}} x^{2\beta}(1-x^2)^{-1/2} H(1-x) \tag{5.19}$$

Also,

$$\int_0^\infty t^{3\alpha+\beta+\theta} J_0(t) k_4(xt)\,dt = \frac{2^{3\alpha+\beta+\theta}}{\Gamma(-\alpha+\beta-\theta)} x^{2\alpha}(1-x^2)^{-(3\alpha+\beta+\theta)} H(1-x),$$

$$\alpha-\beta+\theta < 0, \theta \ge -1. \tag{5.20}$$

Let $\theta = -1$, we get [4, p.48(7)],

$$\int_0^\infty t^{\alpha-\beta} J_1(t) J_{\alpha-\beta}(xt)\,dt = \frac{2^{\alpha-\beta}}{\Gamma(\alpha+3\beta)} x^{\alpha-\beta}(1-x^2)^{-(\alpha-\beta)} H(1-x) \tag{5.21}$$

We also have,

$$\frac{2^{1-\theta}}{\Gamma(\theta)}\int_1^\infty t^{2\alpha}(t^2-1)^{\theta-1} k_4(xt)\,dt = x^{\alpha+\beta-\theta} J_{\alpha-\beta+\theta}(x),\ 0 < \theta < (\alpha+2\beta) \tag{5.22}$$

If $\theta=\frac{1}{2}$, then [5, p.102(29)],

$$\sqrt{\frac{2}{\pi}}\int_1^\infty t^{3\alpha+\beta}\left(t^2-1\right)^{-(\alpha+\beta)}Y_{\alpha-\beta}\left(xt\right)dt=x^{-(\alpha+\beta)}J_{2\alpha}\left(x\right) \tag{5.23}$$

Finally,

$$\int_0^\infty \frac{t^{2\alpha+2\theta}}{a^2+t^2}k_4\left(xt\right)dt=a^{\alpha-\beta+2\theta}x^{\alpha+\beta}K_{\alpha-\beta}\left(ax\right),-1<\theta<\alpha+2\beta \tag{5.24}$$

and generally, [9,p.424(2)],

$$\int_0^\infty \frac{t^{2\alpha+2\theta}}{\left(a^2+t^2\right)^{m+1}}k_4\left(xt\right)dt=\frac{(-1)^m}{m!2^m}x^{\alpha+\beta}\left[\frac{1}{a}\frac{d}{da}\right]^m\left[a^{\alpha-\beta+2\theta}K_{\alpha-\beta}\left(ax\right)\right] \tag{5.25}$$

Now letting $\theta=0$ and $\frac{1}{2}$, (5.24) yields respectively [5, p.23(12)], [4, p.99(15)],

$$\int_0^\infty \frac{t^{3\alpha+\theta}}{a^2+t^2}J_{\alpha-\beta}\left(xt\right)dt=a^{\alpha-\beta}K_{\alpha-\beta}\left(ax\right) \tag{5.26}$$

$$\int_0^\infty \frac{t^{5\alpha+3\beta}}{a^2+t^2}Y_{\alpha-\beta}\left(xt\right)dt=a^{3\alpha+\beta}K_{\alpha-\beta}\left(ax\right) \tag{5.27}$$

Remark: All the results, for which we have not given references from the literature, appear to be new. our method, therefore has yielded a large number of new integration formulae.

**6. Applications**

Since the kernels in this paper are also solutions of a Fourth order ordinary differential equation [1], it is expected that our results will find applications in situations which involve such differential equations. One such situation was encountered in [1]. We point out some more below.

If we consider the problem of finding solutions of

$$\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2 u}{\partial t^2}=0,\ \text{in}\ 0<x<\infty,\ t>0 \tag{6.1}$$

or of

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right]^2u+\frac{\partial^2 u}{\partial t^2}=0\ \text{in}\ 0<r<\infty,t>0 \tag{6.2}$$

which solutions are bounded at infinity, and which satisfy the conditions

$$\text{(i)}\ u=\frac{\partial u}{\partial x}=0\ \text{or}\ \left(u=\frac{\partial u}{\partial r}=0\right)\ \text{at}\ x=0\ \text{or}\ \left(r=0\right) \tag{6.3}$$

or the conditions

$$\text{(ii)}\ \frac{\partial^2 u}{\partial x^2}=\frac{\partial^3 u}{\partial x^3}=0\ \text{or}\ \left(\nabla^2u=\frac{\partial}{\partial r}\nabla^2u=0\right)\ \text{at}\ x=0\ \text{or}\ \left(r=0\right) \tag{6.4}$$

respectively, where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$ ,then we encounter the kernels introduced in this paper. If these solutions are subject to the initial conditions

$$u = g_1(\xi) \tag{6.5a}$$

and

$$\frac{\partial u}{\partial t} = g_2(\xi) \tag{6.5b}$$

at $t = 0$ in $0 < \xi < \infty$ , where $\xi$ is either $x$ or $r$ depending upon whether we are dealing with (6.1) or equation (6.2) then the solution is

$$u(\xi,t) = \int_0^\infty \left[ k(\lambda\xi) A(\lambda)\cos(\lambda^2 t) + \frac{B(\lambda)}{\lambda}\sin(\lambda^2 t) \right] d\lambda \tag{6.6}$$

where

$$g_1(\xi) = \int_0^\infty A(\lambda) k(\lambda\xi) d\lambda \tag{6.7}$$

and

$$g_2(\xi) = \int_0^\infty \lambda B(\lambda) k(\lambda\xi) d\lambda \tag{6.8}$$

where $k$ is an appropriate kernel. If $k$ is self conjugate then the solution of equations (6.7) and (6.8) is

$$A(\lambda) = \int_0^\infty g_1(\xi) k(\lambda\xi) d\xi \tag{6.9}$$

$$B(\lambda) = \frac{1}{\lambda}\int_0^\infty g_2(\xi) k(\lambda\xi) d\xi \tag{6.10}$$

and substitution in equation (6.6) gives $u$ . The following cases should be noted:

1. If $\alpha - \beta = \frac{1}{2}$ and the conditions are $u = \frac{\partial u}{\partial x} = 0$ at the origin, then equation (6.6) gives deflection of a vibrating semi-infinite elastic rod which is clamped at one end (the origin) and is subject to the initial conditions (6.5). In this case $k = k_1(x)\frac{1}{\sqrt{\pi}}\left(\sin x - \cos x + e^{-x}\right)$ which is self conjugate.
2. If $\alpha - \beta = \frac{1}{2}$ and the conditions are $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^3 u}{\partial x^3} = 0$ at the origin, then equation (6.6) gives deflection of a vibrating semi-infinite elastic rod which is free at one end (the origin) and is subject to the initial conditions (6.5). In this case $k = k_2(x) = \frac{1}{\sqrt{\pi}}\left(\cos x - \sin x + e^{-x}\right)$ which is self conjugate.
3. If $\alpha - \beta = 0$ and the conditions are $u = \frac{\partial u}{\partial r} = 0$ at $r = 0$, then equation (6.6) gives deflection of a (symmetrically) vibrating infinite elastic plate which is clamped at the origin and is subject to the initial conditions (6.5).

In this case

$$k = \frac{k_1(r)}{\sqrt{r}} = Y_0(r) + \frac{2}{\pi} k_0(r) \tag{6.11}$$

so that equations (6.7) and (6.8) become

$$\sqrt{\xi}\, g_1(\xi) = \int_0^\infty \frac{A(\lambda)}{\sqrt{\lambda}} k_1(\lambda\xi) d\lambda \tag{6.12}$$

and

$$\sqrt{\xi}g_2(\xi)=\int_0^\infty\sqrt{\lambda}B(\lambda)k_1(\lambda\xi)\,d\lambda \tag{6.13}$$

since $k_1$ is self conjugate, these equations are easily inverted and substitution gives $u$.

It is interesting to note that in case (iii), in the case of a vibrating infinite plate clamped at the origin, the vertical force exerted by the clamp on the plate is given by

$$\lim_{r\to 0}\int_0^{2\pi}-\frac{\partial}{\partial r}\left(\nabla^2 u\right)r\,d\theta=8\int_0^\infty\lambda^2\left[A(\lambda)\cos\left(\lambda^2 t\right)+\frac{B(\lambda)}{\lambda}\sin\left(\lambda^2 t\right)\right]d\lambda \tag{6.14}$$

**References**


1. Aggarueda B.D. and Nasim C., Solutions of Ordinary Differential Equation as a class of Fourier kernels, Internet J. Math. and Math. Sci. Vol.13, No.2, (1990), 377-404.
2. Aggarueda B.D. and Nasim C., On dual integral equations arising in problems of bending of anisotropic plates, Internet J. Math. and Math. Sci. Vol.15, No.3, (1992), 553-562.
3. Erdelyi A, et.al. Tables of Integral Transforms, Vol. I, Bateman Manuscript Project, McGraw-Hill, New York (1954).
4. Erdelyi A, et.al. Tables of Integral Transforms, Vol. II, Bateman Manuscript Project , McGraw-Hill, New York (1954).
5. Guinand A.P., Summation formulae and self reciprocal functions (II), Quart. J. Math. 10 (1939), 104-118
6. Prudnikov A.P., Brychkov Y.A. and Marichev V.I., Integrals and series Vol. 2, Gordon Breach Science Publisher, New York.
7. Titchmarsh E.C., An Introduction to the Theory of Integral Transforms, Second Edition, Oxford University Press (1948),
8. Waphare B.B., Iterated Integral Transforms and related identities. (Communicated).
9. Watson G.N., A Treatise on the theory of Bessel Functions, Second Edition, University Press, Cambridge (1966).