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Generalized fractional Hankel type transform

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Abstract

In this paper the fractional generalization of generalized Hankel type transform is studied which is the generalization of generalized Hankel type transform given by Zemanian [6]. Fractional Hankel type transform in the generalized sense is introduced. Some properties of the kernel are discussed and inversion formula for factional Hankel type transform is proved. Generalized operational relations are derived that can be used to solve certain classes of ordinary and partial differential equations. Finally the values of fractional Hankel type transform are obtained for some special functions.

KEY WORDS:

Fractional transform, Fractional Hankel type transform, Inversion formula, Generalized operational relations.

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1. Introduction:

In recent years Fourier analysis is one of the most frequently used tools in signal processing and is used in many other scientific disciplines. In the mathematics like nature a generalization of the Fourier transform known as the fractional Fourier transform was proposed some years ago [1], [3]. Eventhough potentially useful for signal processing applications, the fractional Fourier transform has been independently reinvented by a number of researchers.

It had briefly introduced in [1] the fractional Fourier transform. He discussed the main properties and presented the new results including the fractional Fourier transform. Also represented simple relationship of the fractional Fourier transform with several time frequency representations that supports the interpretation of it as a rotation operator.

Fiona H. Kerr [2] had defined the fractional Hankel transform with parameter a of $f(x)$ denoted by $H_a\, f(x)$. Following [2], we define the fractional Hankel type transform with parameter a of $f(x)$ denoted by $H_a\, f(x)$ perform a linear operation given by the integral transform,

$$[H_a\, f(x)]\,(y) = \int_0^\infty f(x)\, K_a\,(x,y)\, dx,$$

where

$$K_a\,(x,y) = A_{\alpha,\beta,a}\, e^{-\frac{i}{2}(x^2+y^2)\cos\left(\frac{a}{2}\right)} \left(\frac{xy}{\left|sin\frac{a}{2}\right|}\right)^{\alpha+\beta} J_{\alpha-\beta}\left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)$$

$$= \delta\,(x-y)\ \ for\ a = 0\ \ \&\ \ a = 2\pi,$$

and

$$A_{\alpha,\beta,a} = \left|\sin\frac{a}{2}\right|^{-\frac{1}{2}} e^{i\left(\frac{\pi}{2}\hat{a}-\frac{a}{2}\right)(3\alpha+\beta)}, \qquad \hat{a} = sgna,$$

$$f \in L^2\,(R^+), \qquad a \in R, \qquad and\ (\alpha-\beta) > -1.$$

The above fractional Hankel type transform is the generalization of the Hankel type transform

$$H\big(f(x)\big) = \int_0^\infty (xy)^{\alpha+\beta} f(x)\, J_{\alpha-\beta}\,(xy)\, dx \tag{1.1}$$

for the parameter $a = \pi$, the fractional Hankel type transform reduces to the above Hankel type transform.

The present paper is organized as follows. Section 2 presents the fractional Hankel type transform with parameter a in the sense of generalized function and its interpretation as rotation operator. In Section 3, we give some useful properties of kernel. Inversion formula for this transform is derived Section 4. Section 5 lists some operation transform formulae for the fractional Hankel type transform. Some properties of fractional Hankel type transform are proved in Section 6. In Section 7 we give fractional Hankel type transform of some simple functions, lastly Section 8 concludes. We follow the notations and terminology used in Zemanian [6].

2. Fractional Hankel type transform in the generalized sense:

2.1: The testing function space E: An infinitely differentiable complex valued function ψ on R^n belongs to $E\ (R^n)$ or E if for each compact set $P\ \subset\ S_a$ where

$$S_a = \{x\ \in\ R^n, \quad |x|\ \leq a, \quad a > 0\},.$$

$$\rho_{p,k}(\psi) = \operatorname*{Sup}_{x\,\in\,p}|D^k\ \psi\ (x)|\ < \infty\ , k \in N$$

Clearly E is complete and so a Frechet space.

Moreover we say that f is a fractional transformable if it is a member of E' (the dual space of E).

2.2: The generalized fractional Hankel type transform:

It is easily seen that for each $x\ \in\ R^n\ and\ 0 < a < 2\pi,$ the function $K_a\ (x,y) \in E$ as a function of x. Hence the fractional Hankel type transform of $f\ \in\ E'$ can be defined by

$$[H_a f(x)]\ (y) = \ H_a\ (y) = \ \langle f(x),\ K_a\ (x,y)\rangle\ , \tag{2.1}$$

where

$$K_a\ (x,y) = \ A_{\alpha,\beta,a}\ e^{-\frac{i}{2}(x^2+y^2)\cot\left(\frac{a}{2}\right)} \left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)^{\alpha+\beta} J_{\alpha-\beta}\left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)$$

$$= \ \delta\ (x-y)\ \ for\ \ a = 0\ \ \&\ \ a = 2\pi \tag{2.2}$$

and

$$A_{\alpha,\beta,a} = \ \left|\sin\frac{a}{2}\right|^{-\frac{1}{2}}\ e^{i\left(\frac{\pi}{2}\hat{a}-\frac{a}{2}\right)(3\alpha+\beta)}, \qquad \hat{a} = sgn\ a$$

then the right hand side of (2.1) has a meaning as an application of $f\ \in\ E'\ to\ \ K_a\ (x,y)\ \in E$.

3. Properties of Kernel:

The Kernel $K_a\ (x,y)$ given in (2.2) satisfies the following properties,

1. $K_a\ (x,y) = \ K_a\ (y,x)$
2. $K_{-a}\ (x,y) = \ K_a^*\ (x,y)\ ,\ where\ '*'$ denotes the conjugation
3. $K_a\ (-x,y) = \ K_a\ (x,-y)\ ,$
4. For $a = \pi,$ the kernel coincides with the kernel of the Hankel type transform given in [6].
5. $K_a\ (x,0) = 0$
6. $K_a\ (0,y) = 0$
7. $\int_0^\infty K_a\ \ (x,y)\ K_b(y,z)\ dy = \ \ K_{a+b}\ (x,z).$

Proof : First six properties are simple to prove and hence their proofs are omitted. We prove the last property.

$$L.H.S. = \int_0^\infty K_a\ (x,y)\ K_b\ (y,z)\ dy$$

$$= \ A_{\alpha,\beta,a}\ A_{\alpha,\beta,b}\ e^{-\frac{i}{2}\left(x^2\cot\frac{a}{2} + z^2\cot\frac{b}{2}\right)} \left(\frac{xz}{\left|\sin\frac{a}{2}\right|\ \left|\sin\frac{b}{2}\right|}\right)^{\alpha+\beta}$$

$$\times \int_0^\infty y\, e^{-\frac{i}{2}y^2\left(\cot\frac{a}{2}+\cot\frac{b}{2}\right)} J_{\alpha-\beta}\left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right) J_{\alpha-\beta}\left(\frac{xy}{\left|\sin\frac{b}{2}\right|}\right) dy \,. \qquad (3.1)$$

Let us first evaluate

$$\int_0^\infty y\, e^{-iAy^2} J_{\alpha-\beta}(x,y)\, J_{\alpha-\beta}(z,y)\, dy\,,$$

where $A = \left(\cot\frac{a}{2}+\cot\frac{b}{2}\right)$, $x_1 = \frac{x}{\left|\sin\frac{a}{2}\right|}$, $z_1 = \frac{z}{\left|\sin\frac{b}{2}\right|}$.

$$\int_0^\infty y\, e^{-iAy^2} J_{\alpha-\beta}(x,y)\, J_{\alpha-\beta}(z,y)\, dy = z_1^{-(\alpha+\beta)} \left\{ \begin{array}{c} \int_0^\infty (z_1 y)^{\alpha+\beta}\, y^{\alpha+\beta} J_{\alpha-\beta}(x_1 y)\, J_{\alpha-\beta}(z_1 y) \\ \times \cos(Ay^2)dy - i \int_0^\infty (z_1 y)^{\alpha+\beta}\, y^{\alpha+\beta} J_{\alpha-\beta}(x_1 y)\, J_{\alpha-\beta}(z_1 y) \\ \times \sin(Ay^2)\, dy \end{array} \right\}$$

$$= \frac{1}{\left(\cot\frac{a}{2}+\cot\frac{b}{2}\right)} J_{\alpha-\beta}\left(\frac{xz}{\left|\sin\frac{a}{2}\right|\left|\sin\frac{b}{2}\right|\left(\cot\frac{a}{2}+\cot\frac{b}{2}\right)}\right)$$
$$\times\, e^{i\left[\frac{z^2\sin^2\frac{a}{2}+x^2\sin^2\frac{b}{2}}{\left(\sin^2\frac{a}{2}+\sin^2\frac{b}{2}\right)2\left(\cot\frac{a}{2}+\cot\frac{b}{2}\right)}-(3\alpha+\beta)\frac{\pi}{2}\right]} .$$

Equation (3.1) gives,

$$L.H.S = A_{\alpha,\beta,a}\, A_{\alpha,\beta,b} \left(\frac{xz}{\left|\sin\frac{a}{2}\sin\frac{b}{2}\right|\left(\cot\frac{a}{2}+\cot\frac{b}{2}\right)}\right)^{\alpha+\beta} e^{-\frac{i}{2}\left(x^2\cot\frac{a}{2}+z^2\cot\frac{b}{2}\right)}$$
$$\times \frac{1}{\left(\cot\frac{a}{2}+\cot\frac{b}{2}\right)} J_{\alpha-\beta}\left(\frac{xz}{\left|\sin\frac{a}{2}\sin\frac{b}{2}\right|\left(\cot\frac{a}{2}+\cot\frac{b}{2}\right)}\right)$$
$$\times\, e^{i}\left[\frac{z^2\sin^2\frac{a}{2}+x^2\sin^2\frac{b}{2}}{\left(\sin^2\frac{a}{2}+\sin^2\frac{b}{2}\right)2\left(\cot\frac{a}{2}+\cot\frac{b}{2}\right)}-(3\alpha+\beta)\frac{\pi}{2}\right]$$

After some straight forward steps, we obtain,

$$L.H.S. = \left|\sin\left(\frac{a+b}{2}\right)\right|^{-(\alpha+\beta)} e^{i\left(\frac{\pi}{2}(\hat{a}+\hat{b}-1)-\left(\frac{a+b}{2}\right)\right)}$$
$$\times J_{\alpha-\beta}\left(\frac{xz}{\left|\sin\left(\frac{a+b}{2}\right)\right|}\right) e^{-\frac{i}{2}(x^2+z^2)\cot\left(\frac{a+b}{2}\right)} (3\alpha+\beta)\left(\frac{xz}{\left|\sin\left(\frac{a+b}{2}\right)\right|}\right)^{\alpha+\beta}$$
$$= A_{\alpha,\beta,a+b}\, e^{-\frac{i}{2}(x^2+z^2)\cot\left(\frac{a+b}{2}\right)}\left(\frac{xz}{\left|\sin\left(\frac{a+b}{2}\right)\right|}\right)^{\alpha+\beta} J_{\alpha-\beta}\left(\frac{xz}{\left|\sin\left(\frac{a+b}{2}\right)\right|}\right)$$
$$= K_{a+b}(x,z) = R.H.S.$$

4. Inversion formula:

In this section we derive inversion formula for generalized fractional Hankel type transformation.

It is possible to recover the function f by means of the inversion formula.

$$f(x) = \int_0^\infty \overline{K_a(x,y)}\, H_a(y)\, dy,$$

where

$$\overline{K_a(x,y)} = \bar{A}_{\alpha,\beta,a}\, e^{\frac{i}{2}(x^2+y^2)\cot\left(\frac{a}{2}\right)} \cdot \left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)^{\alpha+\beta} J_{\alpha-\beta}\left(\frac{xy}{\left|sin\frac{a}{2}\right|}\right)$$

and $\overline{K_a(x,y)}$ is complex conjugate of $K_a(x,y)$.

Proof: The one dimensional fractional Hankel type transform is given by

$$[H_a\, f(x)]\,(y) = H_a(y) = \int_0^\infty K_a\,(x,y)\, f(x) dx \qquad (4.1)$$

where the Kernel,

$$K_a\,(x,y) = \left|sin\frac{a}{2}\right|^{\alpha+\beta} e^{i\left(\frac{\pi}{2}\hat{a}-\frac{a}{2}\right)(3\alpha+\beta)} e^{-\frac{i}{2}(x^2+y^2)\cot\frac{a}{2}}$$

$$\times J_{\alpha-\beta}\left(\frac{xy}{\left|sin\frac{a}{2}\right|}\right)\left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)^{\alpha+\beta}$$

$$= A_{\alpha,\beta,a}\, e^{-\frac{i}{2}(x^2+y^2)\cot\frac{a}{2}} J_{\alpha-\beta}\left(\frac{xy}{\left|sin\frac{a}{2}\right|}\right)\left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)^{\alpha+\beta}$$

$$= A_{\alpha,\beta,a}\; e^{\frac{i}{2}(x^2+y^2)\,C_{2a}}\; J_{\alpha-\beta}\,(C_{1a}\,xy)\,(C_{1a}xy)^{\alpha+\beta},$$

where $C_{1a} = \frac{1}{\left|\sin\frac{a}{2}\right|}$, $C_{2a} = -\cot\frac{a}{2}$,

therefore from (4.1),

$$e^{-\frac{i}{2}y^2\,C_{2a}} H_a\,(y) = \int_0^\infty g(x)\, J_{\alpha-\beta}\;(C_{1a}\,xy)\,(C_{1a}\,xy)^{\alpha+\beta}\, dx$$

$= H_a\,[g(x)]\,(\eta)$, where $\eta = C_{1a}\, y$

$= G(\eta)$ (say), where $g(x) = A_{\alpha,\beta,a}\, e^{\frac{i}{2}x^2\,C_{2a}} f(x)$ (4.2)

$e^{-\frac{i}{2}y^2\,C_{2a}} H_a\,(y) = H_a\,[g(x)]\,(\eta) = G(\eta)$ is the Hankel transform of $g(x)$ with argument η.

Involving Hankel inversion we can write,

$$g(x) = \int_0^\infty G(\eta)\, J_{\alpha-\beta}\,(x\eta)(x\eta)^{\alpha+\beta} d\eta\;,$$

where

$$G(\eta) = e^{-\frac{i}{2}C_{2a}\left(\frac{\eta}{C_{1a}}\right)^2} H_a\left(\frac{\eta}{C_{1a}}\right).$$

Putting the value of $g(x)$ from (4.2) and on simplifying, we get inversion formulae,

$f(x) = \int_0^\infty \overline{K_a(x/y)}\, H_a\,(y)\, dy$, where

$$\overline{K_a(x,y)} = \overline{A_{\alpha,\beta,a}}\, e^{\frac{i}{2}(x^2+y^2)\,cot\frac{a}{2}} J_{\alpha-\beta}\left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)\left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)^{\alpha+\beta}$$

and

$$\overline{A_{\alpha,\beta,a}} = \left|\sin\frac{a}{2}\right|^{\alpha+\beta} e^{-i\left(\frac{a}{2}-\frac{\pi}{2}\hat{a}\right)(3\alpha+\beta)}.$$

5. Generalized operational relation of fractional Hankel type transform:

As is well known an operational calculus can be used on the usual Hankel type transform, we derive an operational relations involving first derivatives and operational relations having second derivatives.

5.1: Operational relations involving first derivatives:

We derive operational transform involving $H_a\left(\frac{df}{dx}\right)$ obtained by inserting $\frac{df}{dx}$, instead of $f(x)$ in the integral representation (2.1), we then integrate by parts,

$$\left[H_a\left(\frac{df}{dx}\right)\right](y) = A_{\alpha,\beta,a}\, e^{-\frac{i}{2}y^2 cot\frac{a}{2}} \int_0^\infty e^{-\frac{i}{2}cot\,\frac{a}{2}} \left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)^{\alpha+\beta} J_{\alpha-\beta}\left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)\frac{df}{dx}\,dx$$

$$= -A_{\alpha,\beta,a}\, e^{-\frac{i}{2}y^2 cot\frac{a}{2}} \left\{ \begin{array}{c} \left(\frac{y}{\left|\sin\frac{a}{2}\right|}\right)^{\alpha+\beta} \int_0^\infty e^{-\frac{i}{2}x^2 cot\,\frac{a}{2}}\, x^{\alpha+\beta} J_{\alpha-\beta}\left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right) \\ \times f dx + \int_0^\infty e^{-\frac{i}{2}x^2 cot\,\frac{a}{2}} \left(-ix\,cot\,\frac{a}{2}\right)\left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)^{\alpha+\beta} J_{\alpha-\beta}\left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right) f dx \\ + \int_0^\infty e^{-\frac{i}{2}x^2 cot\,\frac{a}{2}} \left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)^{\alpha+\beta} \left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right) J_{\alpha-\beta}\left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)\frac{f}{x}\,dx \end{array} \right\}$$

$$= -H_a\left(\frac{f}{2x}\right) + i\,cot\,\frac{a}{2}\,H_a(xf) + (\alpha-\beta)\,H_a\left(\frac{f}{x}\right) - A_{\alpha,\beta,a}\, e^{-\frac{i}{2}y^2 cot\,\frac{a}{2}} \int_0^\infty e^{-\frac{i}{2}x^2 cot\,\frac{a}{2}} \left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)^{\alpha+\beta}\left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)$$

$$\times\, J_{\alpha-\beta-1}\left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)\frac{f}{x}\,dx\,.$$

After some straight forward steps, we obtain,

$$H_a\left(\frac{df}{dx}\right) = -\frac{1}{2}\,H_a\left(\frac{f}{x}\right) + i\,cot\,\frac{a}{2}\,H_a(xf) - \frac{y}{\left|\sin\frac{a}{2}\right|}\,H_a(f) \qquad (5.1)$$

is the operational relation involving first derivative in (5.1) replace f by fx , $\frac{df}{dx} \to f + \frac{df}{dx}$ gives

$$H_a\left(x\,\frac{df}{dx}\right) = -\frac{3}{2}\,H_a(f) + i\,cot\,\frac{a}{2}\,H_a(x^2 f) - \frac{y}{\left|\sin\frac{a}{2}\right|}\,H_a(xf). \qquad (5.2)$$

Replacing f by $\frac{f}{x}$ in (2.2),

$$\frac{df}{dx} \to \frac{1}{x}\frac{df}{dx} - \frac{f}{x^2}$$

gives

$$H_a\left(\frac{1}{x}\frac{df}{dx}\right) = \frac{1}{2}\,H_a\left(\frac{f}{x^2}\right) + i\,cot\,\frac{a}{2}\,H_a(f) - \frac{y}{\left|\sin\frac{a}{2}\right|}\,H_a\left(\frac{f}{x}\right). \qquad (5.3)$$

5.2 : Operational relations involving second derivatives:

We now calculate second derivative $H_a\left(\frac{d^2f}{dx^2}\right)$ by inserting $\frac{df}{dx}$ in place of f in the equation (5.1), we obtain

$$H_a\left(\frac{d^2f}{dx^2}\right) = -\frac{1}{2}\,H_a\left(\frac{1}{x}\frac{df}{dx}\right) + i\cot\frac{a}{2}\,H_a\left(x\,\frac{df}{dx}\right) - \frac{y}{\left|\sin\frac{a}{2}\right|}\,H_a\left(\frac{df}{dx}\right)$$

$$= \frac{1}{4}\,H_a\left(\frac{f}{x^2}\right) + \frac{y}{\left|\sin\frac{a}{2}\right|}\,H_a\left(\frac{f}{x}\right) - \frac{2yi\cot\frac{a}{2}}{\left|\sin\frac{a}{2}\right|}\,H_a(fx)$$

$$-cot^2\left(\frac{a}{2}\right)H_a(fx^2) + \left(\frac{y^2}{sin^2\frac{a}{2}} - \frac{i}{2}\cot\frac{a}{2}\right)H_a(f)\,.$$

6. Properties of fractional Hankel type transform:

We prove the following properties of fractional Hankel type transform.

I. $H_a[f(x)] = \frac{A_{\alpha,\beta,a}}{C\,A_{\alpha,\beta,b}}\, e^{\left[-\frac{i}{2}\left(\frac{y^2}{c^2}\frac{\left|sin\frac{b}{2}\right|^2}{\left|\sin\frac{a}{2}\right|^2}\right)\left(c^4\frac{\left|sin\frac{a}{2}\right|^2}{\left|sin\frac{b}{2}\right|^2} - \frac{1}{c^2}\right)\cot\frac{b}{2}\right]}$

$$\times\ H_b\, f(x)\left(\frac{y}{c}\frac{\left|\sin\frac{b}{2}\right|}{\left|\sin\frac{a}{2}\right|}\right),$$

where $\cot\frac{b}{2} = \frac{1}{c^2}\cot\frac{a}{2}$.

Proof:

$$H_a\,[f(x)] = \int_0^\infty f(x)\, A_{\alpha,\beta,a}\, e^{-\frac{i}{2}(x^2+y^2)\cot\frac{a}{2}} \left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)^{\alpha+\beta}$$

$$\times\ J_{\alpha-\beta}\left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right) dx$$

$$= \int_0^\infty f(t)\, A_{\alpha,\beta,a}\, e^{-\frac{i}{2}\left(\frac{t^2}{c^2}+y^2\right)\cot\frac{a}{2}} \left(\frac{ty}{c\left|\sin\frac{a}{2}\right|}\right)^{\alpha+\beta} J_{\alpha-\beta}\left(\frac{ty}{c\left|sin\frac{a}{2}\right|}\right)\frac{dt}{c},$$

where $t = cx$,

$$= \frac{A_{\alpha,\beta,a}}{c\,A_{\alpha,\beta,b}}\int_0^\infty f(t)\ A_{\alpha,\beta,b}\, e^{-\frac{i}{2}(t^2+c^2y^2+z^2-z^2)\cot\frac{b}{2}}$$

$$\times\left(\frac{tz}{\left|sin\frac{b}{2}\right|}\right)^{\alpha+\beta} J_{\alpha-\beta}\left(\frac{tz}{\left|sin\frac{b}{2}\right|}\right) dt\,,\qquad where\ \ z = \frac{y\ \left|\sin\frac{b}{2}\right|}{c\ \left|\sin\frac{a}{2}\right|}$$

$$= \frac{A_{\alpha,\beta,a}}{c\,A_{\alpha,\beta,b}}\, e^{-\frac{i}{2}z^2\left(\frac{c^4\left|\sin\frac{a}{2}\right|^2}{\left|\sin\frac{b}{2}\right|^2}-\frac{1}{c^2}\right)\cot\frac{b}{2}} H_b\,[f(x)]\left(\frac{y\left|\sin\frac{b}{2}\right|}{c\ \left|\sin\frac{a}{2}\right|}\right),$$

$$= \frac{A_{\alpha,\beta,a}}{c\,A_{\alpha,\beta,b}}\, e^{-\frac{i}{2}\left(\frac{y^2\left|\sin\frac{b}{2}\right|^2}{c^2\left|\sin\frac{a}{2}\right|^2}\right)\left(\frac{c^4\left|\sin\frac{a}{2}\right|^2}{\left|\sin\frac{b}{2}\right|^2}-\frac{1}{c^2}\right)\cot\frac{b}{2}} H_b\,[f(x)]\left(\frac{y\ \left|\sin\frac{b}{2}\right|}{c\ \left|\sin\frac{a}{2}\right|}\right).$$

II. $H_a\left(e^{\frac{i}{2}x^2}f(x)\right) = A_{\alpha,\beta,a}\, e^{-\frac{i}{2}y^2\cot\, a}\, H_a\left(f(x)\right)\left(\frac{y}{|\sin\ a|}\right)$.

Proof of the above result is simple and hence omitted.

7. Transform of some common functions:

The Hankel type transform of some common functions are proved.

Result 1:

$$\delta\,(x-\tau) = \left|\sin\frac{a}{2}\right|^{-(\alpha+\beta)} e^{i\left(\frac{a}{2}-\frac{\pi}{2}\hat{a}\right)(3\alpha+\beta)-\frac{i}{2}(\tau^2+y^2)\cot\frac{a}{2}}$$

$$\times\ J_{\alpha-\beta}\left(\frac{\tau y}{\left|\sin\frac{a}{2}\right|}\right)\left(\frac{\tau y}{\left|\sin\frac{a}{2}\right|}\right)^{\alpha+\beta}.$$

Proof is simple and hence omitted.

Result 2:

$$H_a\left[x^{2\alpha}\,e^{-nx^2}\right] = A_{\alpha,\beta,a}\, e^{-\frac{i}{2}y^2\cot\frac{a}{2}}\left[\frac{\left(\frac{y}{\left|\sin\frac{a}{2}\right|}\right)^{2\alpha}}{(2n)^{3\alpha+\beta}}\, e^{-\left(\frac{y}{\left|sin\frac{a}{2}\right|}\right)^2/8n}\right],$$

$$Re\ n\ > 0,\qquad Re\ (\alpha-\beta) > -1.$$

Proof:

$$H_a\left[x^{2\alpha}\,e^{-nx^2}\right] = A_{\alpha,\beta,a}\,e^{-\frac{i}{2}y^2\cot\frac{a}{2}}\int_0^\infty e^{-\frac{i}{2}x^2\cot\frac{a}{2}}\,x^{2\alpha}\,e^{-nx^2}$$
$$\times\ (xy_1)^{\alpha+\beta}\,J_{\alpha-\beta}\,(xy_1)\,dx\,,$$

where

$$y_1 = \frac{y}{\left|\sin\frac{a}{2}\right|}\,,$$

$$= A_{\alpha,\beta,a}\ e^{-\frac{i}{2}y^2\cot\frac{a}{2}}\int_0^\infty x^{2\alpha}\,e^{-Bx^2}(xy_1)^{\alpha+\beta}\,J_{\alpha-\beta}\,(xy_1)\,dx\,,$$

where

$$B = n+\frac{i}{2}\cot\frac{a}{2}$$

$$= A_{\alpha,\beta,a}\ e^{-\frac{i}{2}y^2\cot\frac{a}{2}}\left(\frac{\left(\frac{y}{\left|\sin\frac{a}{2}\right|}\right)^{2\alpha}}{(2n)^{2\alpha+\beta}}.\,e^{-\left(\frac{y}{\left|\sin\frac{a}{2}\right|}\right)^2/8n}\right)$$

Result 3:

$$H_a\left[J_{\alpha-\beta}\,(nx)\,x^{\alpha+\beta}\right] = A_{\alpha,\beta,a}\ e^{-\frac{i}{2}y^2\cot\frac{a}{2}}\int_0^\infty e^{-\frac{i}{2}x^2\cot\frac{a}{2}}\,x^{\alpha+\beta}\,J_{\alpha-\beta}(ax)$$
$$\times\left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)^{\alpha+\beta} J_{\alpha-\beta}\left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)dx$$

Proof:

$$H_a\left[J_{\alpha-\beta}(nx)\,x^{\alpha+\beta}\right] = A_{\alpha,\beta,a}\,e^{-\frac{i}{2}y^2\cot\frac{a}{2}}\int_0^\infty e^{-\frac{i}{2}x^2\cot\frac{a}{2}}\,x^{\alpha+\beta}\,J_{\alpha-\beta}\,(nx)$$
$$\times\left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)^{\alpha+\beta} J_{\alpha-\beta}\left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)dx\,.$$

$$\frac{y}{\left|\sin\frac{a}{2}\right|}\,A_{\alpha,\beta,a}\left[\begin{array}{c}\frac{1}{y^{\alpha+\beta}}\int_0^\infty x^{\alpha+\beta}\,(xy)^{\alpha+\beta}\cos\left(\frac{1}{2}\cot\frac{a}{2}\,x^2\right)J_{\alpha-\beta}\,(nx)\\ \times\,J_{\alpha-\beta}\left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)dx\\ -i\,\frac{1}{y^{\alpha+\beta}}\int_0^\infty x^{\alpha+\beta}\,(xy)^{\alpha+\beta}\sin\left(\frac{1}{2}\cot\frac{a}{2}\,x^2\right)J_{\alpha-\beta}\,(nx)\\ \times\,J_{\alpha-\beta}\left(\frac{xy}{\left|\sin\frac{a}{2}\right|}\right)dx\end{array}\right]$$

$$= \frac{A_{\alpha,\beta,a}}{\left|\sin\frac{a}{2}\right|}\cdot\frac{y^{\alpha+\beta}}{\cot\left(\frac{a}{2}\right)}\,J_{\alpha-\beta}\left(\frac{ny}{\cot\frac{a}{2}}\right)e^{-i\left(\frac{a^2-\left(\frac{y}{\left|\sin\frac{a}{2}\right|}\right)^2}{2\cot\frac{a}{2}}-\frac{(\alpha-\beta)\,\pi}{2}\right)}$$

Result 4:

$$H_a\left[x^\lambda e^{\frac{i}{2}x^2\cot\frac{a}{2}}\right] = A_{\alpha,\beta,a}\,.\,e^{-\frac{i}{2}y^2\cot\frac{a}{2}}\,.\,2^{\lambda+\frac{1}{2}} \times\left(\frac{y}{\left|\sin\frac{a}{2}\right|}\right)^{-\lambda-1}\frac{\Gamma\left(\frac{\lambda}{2}+\frac{3\alpha+\beta}{2}\right)}{\Gamma\left(\alpha-\frac{\lambda}{2}\right)}.$$

Proof of the above result is simple and hence is omitted.

8. Conclusion :

We have introduced an extension of Hankel type transform that is designated fractional Hankel type transform. This linear transform depends on a parameter a and can be interpreted as a rotation by an angle a. When $a=\pi$, the fractional Hankel type transform coincides with the conventional Hankel type transform. Inversion formula for this transform is also established. We derive an operational relation for first and second order derivative for fractional Hankel type transform. Some properties of the fractional Hankel type transform are given which coincides with corresponding properties for Hankel type transform in special case. Fractional Hankel type transform of some simple functions are also obtained.

Remarks:

1. If we set $\alpha=\frac{1}{4}+\frac{\nu}{2}$, $\beta=\frac{1}{4}-\frac{\nu}{2}$ throughout this paper, it reduces to results in [5].
2. Author claims that the results in the present paper are more general than that of [5].

References:

1. L.B. Almeida; An introduction to the angular Fourier transform, in proc. 1993, IEEE Int. Conf. Acoust. Speech, signal processing (Minneapolis, MN), April, 1993.
2. Fiona H. Kerr, A fractional power theory for Hankel transform, Int. J. Mathematical Analysis and Applications 158, 114-123 (1991).
3. V. Namias, The fractional order Fourier transform and its application to quantum mechanics, J. Inst. Math. Appl., Vol. 25, 241-265, 1980.
4. V. Namias, Fractionalization of Hankel transform, J. Inst., Math. Appl., Vol.26, 187-197, 1980.
5. R.D. Taywade, A.S. Gudadhe and V.N. Mahale, On generalized fractional Hankel transform, Internat. J. Math. Anal., Vol. 6, 2012, no. 18, 883-896.
6. A.H. Zemanian, A generalized integral transformation, Interscience Publishers, New York, 1968.