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Hankel type transform on a Gevrey type space

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Abstract:

In this paper we give a completion for the distributional theory of Hankel type transformation developed in [6] and [8]. The space H_w generalizing the Altenburg space H is defined. Some properties of this space are studied. It is shown that the Hankel type transformation $h_{\alpha,\beta}$ is an automorphism of H_w. The generalized Hankel type transform of Gevrey spaces is defined and it is found that Hankel transform is an automorphism of H'_w. Product on H_w, Hankel type translation and Hankel type convolution on H_w are investigated.

KEY WORDS:

Hankel type transform,

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1. Introduction : The space $L^p_{\alpha,\beta}$, $1 \le p < \infty$, consists of all those measurable functions ϕ on $I = (0, \infty)$ such that

$$\|\phi\|_{L^p_{\alpha,\beta}} = \left(\int_0^\infty |\phi(x)|^p \, dv(x)\right)^{1/p} < \infty, \quad (1.1)$$

where $L^\infty_{\alpha,\beta}$ denotes the space of those functions ϕ for which

$$\|\phi\|_{L^p_{\alpha,\beta}} = \underset{x \in I}{\text{ess } \ell.\text{u. b}} \, |\phi(x)| < \infty. \quad (1.2)$$

Note that

$$dv(y) = \frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)} \, dy \, . \quad (1.3)$$

The Hankel type transformation of $\phi \in L^1_{\alpha,\beta}(I)$ is defined by

$$\left(h_{\alpha,\beta} \, \phi\right)(x) = \int_0^\infty j_{\alpha-\beta}(xy) \, \phi(y) \, dv(y), \; x \in I, \quad (1.4)$$

where

$j_{\alpha-\beta}(x) = x^{-(\alpha-\beta)} J_{\alpha-\beta}(x)$ and $J_{\alpha-\beta}(x)$ represent the Bessel type function of the first kind and order $(\alpha - \beta)$.

Throughout this paper we shall assume that $(\alpha - \beta) \ge -1/2$.

Since $z^{-(\alpha-\beta)} J_{\alpha-\beta}(z)$ is bounded on I, the Hankel type transform $h_{\alpha,\beta}(\phi)$ is bounded on I. The inversion for (1.4) is given by

$$\phi(y) = \int_0^\infty j_{\alpha-\beta}(xy) \left(h_{\alpha,\beta}\phi\right)(x) \, dv(x), \; y \in I. \quad (1.5)$$

G. Altenburg [1] introduced the space H that consist of all those complex valued and smooth function ϕ defined on I, such that for every $m, n \in \mathbb{N}_0$.

$$\rho_{m,n}(\phi) = \underset{x \in I}{Sup} \, (1 + x^2)^m \left|(x^{-1} D)^n \phi(x)\right| < \infty, \quad (1.6)$$

when H is endowed with the topology associated with the family $\{\rho_{m,n}\}_{m,n \in \mathbb{N}_0}$ of seminorms, H is a Frechet space and the Hankel type transformation $h_{\alpha,\beta}$ is an automorphism of H, [1]. The Hankel type transform is defined on H', the dual space of H, as the transpose of $h_{\alpha,\beta}$ on H and is defined by $h'_{\alpha,\beta}$. That is if $f \in H'$, then the Hankel type transform $h'_{\alpha,\beta} f$ of f is the element of H' defined by

$$\langle h'_{\alpha,\beta} f, \phi \rangle = \langle f, h_{\alpha,\beta} \, \phi \rangle, \qquad \phi \in H.$$

If $f \in L^p_{\alpha,\beta}$ then f defines an element of H' through

$$\langle f, g \rangle = \int_0^\infty f(x) \, g(x) \, dv(x), \; g \in H \, . \quad (1.7)$$

Thus, $L^p_{\alpha,\beta}$ can be seen as a subspace of H'. The convolution associated to the $h_{\alpha,\beta}-$ transformation is defined as follows:

The Hankel type convolution $f \#_{\alpha,\beta}\, g$ of order $(\alpha-\beta)$ of the measurable functions $f, g \in L^1_{\alpha,\beta(I)}$ is given through

$$\left(f \#_{\alpha,\beta}\, g\right)(x) = \int_0^\infty f(y)\left({}_{\alpha,\beta}\tau_x\ g\right)(y)\, d\nu\,(y)\,, \tag{1.8}$$

where the Hankel type translation operator ${}_{\alpha,\beta}\tau_x\ g$ of g is defined by

$$\left({}_{\alpha,\beta}\tau_x\ g\right)(x) = \int_0^\infty g(z)\, D_{\alpha,\beta}\,(x,y,z)\, d\nu\,(z) \tag{1.9}$$

provided that the above integrals exist. Here $D_{\alpha,\beta}$ is the following function

$$D_{\alpha,\beta}\,(x,y,z) = \int_0^\infty j_{\alpha-\beta}\,(xt)\ j_{\alpha-\beta}\,(yt)\, j_{\alpha-\beta}\,(zt)\, d\nu\,(t) \tag{1.10}$$

and we have the following basic formula

$$\int_0^\infty j_{\alpha-\beta}\,(zt)\, D_{\alpha,\beta}\,(x,y,z)\, d\nu\,(z) = j_{\alpha-\beta}\,(xt)\, j_{\alpha-\beta}\,(yt)\,. \tag{1.11}$$

Lemma 1.1: Let f and g be functions on $L^1_{\alpha,\beta}\,(I)$, then

(i) $\left(h_{\alpha,\beta}\left({}_{\alpha,\beta}\tau_x f\right)\right)(y) = j_{\alpha-\beta}\,(xy)\left(h_{\alpha,\beta}\, f\right)(y)$,

(ii) $\left(h_{\alpha,\beta}\left(f \#_{\alpha,\beta}\, g\right)\right)(y) = \left(h_{\alpha,\beta}\, f\right)(y)\left(h_{\alpha,\beta}\, g\right)(y)\,.$

Proof: (i) As $\left({}_{\alpha,\beta}\tau_x f\right)(y) = \int_0^\infty f\,(z)\, D_{\alpha,\beta}\,(x,y,z)\, d\nu\,(z)\,.$

We therefore have

$$\left(h_{\alpha,\beta}\left({}_{\alpha,\beta}\tau_x f\right)\right)(y) = \int_0^\infty j_{\alpha-\beta}\,(yt)\left({}_{\alpha,\beta}\tau_x f\right)(t)\, d\nu\,(t)$$

$$= \int_0^\infty j_{\alpha-\beta}\ (yt)\, d\nu\,(t) \int_0^\infty f\,(z)\, D_{\alpha,\beta}\,(x,t,z)\, d\nu\,(z)$$

$$= \int_0^\infty f(z)\, d\nu\ (z) \int_0^\infty j_{\alpha-\beta}\ (yt)\, D_{\alpha,\beta}\ (x,t,z)\, d\nu\,(t).$$

Using (1.11), we have

$$\left(h_{\alpha,\beta}\left({}_{\alpha,\beta}\tau_x f\right)\right)(y) = \int_0^\infty f(z)\ j_{\alpha-\beta}\,(xy)\, j_{\alpha-\beta}\,(zy)\, d\nu\,(z)$$

$$= j_{\alpha-\beta}\,(xy) \int_0^\infty j_{\alpha-\beta}\,(zy)\, f(z)\, d\nu\,(z)$$

$$= j_{\alpha-\beta}(xy)\left(h_{\alpha,\beta}f\right)(y). \qquad (1.12)$$

(ii) Proof follows from [5, p.339].

Thus proof is completed.

2. The Space H_w : Following G. Bjorck [4], we consider continuous, increasing and non-negative functions w defined on I, such that $w(0) = 0$, $w(x) > 0$ and it satisfies the following three properties :

$$\text{(i) } w(x+y) \leq w(x) + w(y), \; x, y \in I \qquad (2.1)$$

$$\text{(ii)} \int_0^\infty \frac{w(x)}{1+x^2}\, dx < \infty \qquad (2.2)$$

and

(iii) $w(x) \geq a + b\log(1+x)$, for some $a \in R$ and $b > 0$. (2.3)

The class of all such w functions is denoted by M. Note that if w is extended to $\mathbb{R}$ as an even function then w satisfies the subadditivity property

(i) for every $x, y \in \mathbb{R}$..

A. Beurling [3] developed the foundations of a general theory of distributions that extends the Schwartz theory. Some aspects of that theory were presented and completed by Bjorck [4]. Now we collect some definitions and properties for the purpose of the present paper.

A function $\phi \in L^1_{\alpha,\beta}(I)$ is in H_w when ϕ is smooth function and every , $n \in \mathbb{N}_0$,

$$\eta_{m,n}(\phi) = \operatorname*{Sup}_{x \in I} e^{m\,w(x)} \left|(x^{-1}D)^n \phi(x)\right| < \infty . \qquad (2.4)$$

On H_w we consider the topology generated by the family $\{\eta_{m,n}\}_{m,n \in \mathbb{N}_0}$ of seminorms. H_w is clearly a linear space. Following [2], we conclude that H_w is a Frechet space. For,

$w(x) = \log(1+x^2)$ it reduces to H and for $w(x) = x^p (0 < p < 1)$, H_w is a Gevrey space. From definitions (1.6), (2.4) and the inequality $(1+x^2) \leq e^{w(x)}$, it follows that $H_w \subseteq$ H. It is clear that $D(I) \subset H_w(I) \subset E(I)$. Since $D(I)$ is a dense subspace of $E(I)$, then $H_w(I)$ is dense in $E(I)$. Hence $E'(I) \subset H'_w(I)$ the dual of $H_w(I)$. Since $H_w \subset H$ the following properties given in [2,5] hold in the present case also when $w \in M$. The Bessel type operator defined by

$$\Delta_{\alpha,\beta} = x^{-4\alpha} D\, x^{4\alpha} D = D^2 + \frac{4\alpha}{x} D .$$

By an application of Bessels equation for t fixed, we have

$$\Delta_{\alpha,\beta}\, j_{\alpha-\beta}(xt) = -t^2\, j_{\alpha-\beta}(xt).$$

Theorem 2.1: (i) The Hankel type transformation $h_{\alpha,\beta}$ is an automorphism of H_w.

(ii) The generalized Hankel type transformation $h'_{\alpha,\beta}$ is an automorphism of H'_w.

Proof: Here we prove (i) first and (ii) can be proved in a similar way.

As Hankel type transformation $H_{\alpha,\beta}$ is defined by

$$\left(H_{\alpha,\beta}\,\phi\right)(x) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\,\phi(y)\,dy, \quad x \in I, \tag{2.5}$$

we have

$$\left(h_{\alpha,\beta}\,\phi\right)(x) = \int_0^\infty j_{\alpha-\beta}(xy)\,\phi(y)\,d\nu(y)$$

$$= \frac{1}{2^{\alpha-\beta}\,\Gamma(3\alpha+\beta)} \int_0^\infty (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy)\,\phi(y)\,y^{4\alpha}\,dy$$

$$= \frac{1}{2^{\alpha-\beta}\,\Gamma(3\alpha+\beta)} \int_0^\infty (xy)^{(\alpha+\beta)} J_{\alpha-\beta}(xy)\,(xy)^{2\beta-1}\,\phi(y)\,y^{4\alpha}\,dy$$

$$= \frac{1}{2^{\alpha-\beta}\,\Gamma(3\alpha+\beta)}\, x^{2\beta-1} \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\,(y^{2\alpha}\,\phi)(y)\,dy$$

$$= \frac{1}{2^{\alpha-\beta}\,\Gamma(3\alpha+\beta)}\, x^{2\beta-1}\, H_{\alpha,\beta}\,(y^{2\alpha}\phi)(x) \tag{2.6}$$

Using relation (2.6), we have

$$(1+x^2)^m\,(x^{-1}D)^n\left(h_{\alpha,\beta}\,\phi\right)(x)$$

$$= \frac{1}{2^{\alpha-\beta}\,\Gamma(3\alpha+\beta)}\,(1+x^2)^m\,(x^{-1}D)^n\,x^{2\beta-1}\,H_{\alpha,\beta}\,(y^{2\alpha}\phi)\,.$$

Following technique of Zemanian [9, p.141], we can write

$$(1+x^2)^m\left|(x^{-1}D)^n\left(h_{\alpha,\beta}\,\phi\right)(x)\right|$$

$$= \frac{1}{2^{\alpha-\beta}\,\Gamma(3\alpha+\beta)}\left|\int_0^\infty \begin{matrix}(1+y^2)^{2(\alpha-\beta)+m+2n+1}(y^{-1}D)^n\,y^{2\beta-1}\\ \times\left(y^{3\alpha+\beta}\,\phi\right)(y)\left[(xy)^{-(\alpha-\beta)-n}\,J_{\alpha-\beta+m+n}(xy)\right]dy\end{matrix}\right|.$$

From (ii) it follows that to every $\epsilon > 0$, there exist a constant $c(\epsilon)$ such that

$w(\xi) \le \epsilon\,\xi + c(\epsilon)$, so that

$$e^{\nu\,w(\xi)} < e^{\nu\,c(\epsilon)} \sum_{m=0}^{\infty} \frac{(\nu\epsilon)^m}{m!}\,\xi^m \le e^{\nu c(\epsilon)} \sum_{m=0}^{\infty} \frac{(\nu\epsilon)^m}{m!}\,(1+\xi^2)^m.$$

Now, for any choice ν and n , we have

$$\eta_{\nu,n}\left(h_{\alpha,\beta}\,\phi\right) = \sup_{\xi\epsilon I} e^{\nu\,w(\xi)}\left|(\xi^{-1}D)^n\left(h_{\alpha,\beta}\,\phi\right)(\xi)\right|$$

$$\leq \underset{\xi \in I}{Sup}\, e^{\nu\, c(\epsilon)} \sum_{m=0}^{\infty} \frac{(\nu\epsilon)^m}{m!} (1+\xi^2)^m \left|(\xi^{-1}D)^n \left(h_{\alpha,\beta}\, \phi\right)(\xi)\right|$$

$$\leq \underset{\xi \in I}{Sup}\, e^{\nu\, c(\epsilon)} \left| \begin{matrix} \sum_{m=0}^{\infty} \frac{(\nu\epsilon)^m}{m!} \frac{1}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)} \\ \times \int_0^\infty (1+y^2)^{2(\alpha-\beta)+m+2n+1} (y^{-1}D)^m\, \phi(y) \left[(\xi y)^{-(\alpha-\beta)-n} J_{\alpha-\beta+m+n}(\xi y)\, dy\right] \end{matrix} \right|.$$

Since
$\frac{1}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)} (\xi y)^{-(\alpha-\beta)-n} J_{\alpha-\beta+m+n}(\xi y)$ is bounded by $Q_{\alpha,\beta}$, for $(\alpha-\beta) \geq -\frac{1}{2}$, so that

$$\eta_{\nu,n}\left(h_{\alpha,\beta}\phi\right) \leq Q_{\alpha,\beta}\; e^{\nu\, c(\epsilon)} \sum_{m=0}^{\infty} \frac{(\nu\epsilon)^m}{m!} \underset{y \in I}{Sup} (1+y^2)^{2(\alpha-\beta)+m+2n+2}$$

$$\times\ |(y^{-1}\, D)^m\, \phi(y)| \int_0^\infty \frac{dy}{1+y^2}$$

$$\leq Q_{\boldsymbol{\alpha},\boldsymbol{\beta}}\, e^{\nu c(\epsilon)} \sum_{m=0}^{\infty} \frac{(\nu\epsilon)^m}{m!} \gamma_2 \left(\alpha-\beta+{}^{m}/_{2}+n+1), m(\phi)\right) < \infty,$$

as infinite series can be made convergent for $m \geq 1$, by choosing

$\epsilon < \nu^{-1} \left(\gamma_2(\alpha-\beta+m/2+n+1), m(\phi)\right)^{-1/m}$.

This proves that $\left(h_{\alpha,\beta}\phi\right)$ is also in H_w and that $h_{\alpha,\beta}$ is continuous linear mapping from H_w into itself. Since $H_w \subset L^1_{\alpha,\beta}(I)$ for $(\alpha-\beta) \geq -1/2$, we can apply inversion theorem and also the fact that $h^{-1}_{\alpha,\beta} = h_{\alpha,\beta}$ to this case and conclude that $h_{\alpha,\beta}$ is an automorphism on H_w. Thus proof is completed.

Now the generalized Hankel type transformation $h'_{\alpha,\beta}$ on H'_w is defined as adjoint of $h_{\alpha,\beta}$ on H_w. More specifically, for any $\phi \in H_w$ and $\psi \in H'_w$ we have

$\langle h'_{\alpha,\beta}\psi, \phi\rangle = \langle \psi, h_{\alpha,\beta}\, \phi\rangle$.

3. Product, Hankel type translation and Hankel type convolution on I:

We shall denote by Λ_m the space of all C^∞ function $\phi(x)$, $x \in I$, and that $m \in \mathbb{N}_0$ and there exists a $\lambda = \lambda(m) \in \mathbb{N}_0$ for which

$$e^{-\lambda w(x)} |(x^{-1}D)^m \phi(x)| < \infty \tag{3.1}$$

Here Λ is the space of multipliers for H_w.

Theorem 3.1: If $\phi,\ \psi \in H_w(I)$, then $\phi\, \psi \in H_w(I)$.

Proof: For $k, n \in \mathbb{N}_0$, we have by definition (2.4)

$$\eta_{k,n}\,(\phi\psi)\,(x) = \sup_{x\in I} e^{kw(x)}\left|(x^{-1}D)^n\,\phi(x)\,\psi(x)\right|.$$

Using Leibnitz theorem, we have

$\eta_{k,n}\,(\phi\psi)\,(x)$

$$= \sup_{x\in I} e^{k\,w(x)}\left|\sum_{\nu=0}^{n}\binom{n}{\nu}\,(x^{-1}D)^{n-\nu}\,\phi(x)(x^{-1}D)^{\nu}\,\psi(x)\right|$$

$$= \sum_{\nu=0}^{\infty}\binom{n}{\nu}\,\sup_{x\in I} e^{kw(x)}\left|(x^{-1}D)^{n-\nu}\,\phi(x)\right|\,\left|\sup_{x\in I}\,(x^{-1}D)^{\nu}\,\psi(x)\right|$$

$$= \sum_{\nu=0}^{n}\binom{n}{\nu}\,\eta_{k,n-\nu}\,\phi(x)\;\;\eta_{0,\nu}\,\psi(x)\;<\infty.$$

Hence $\phi\,\psi\;\in\;H_w\,(I)$. This completes the proof of theorem.

Theorem 3.2: The mapping $\phi\;\mapsto\;{}_{\alpha,\beta}\tau_x\,\phi$ is continuous from H_w into itself.

Proof: Let $\phi\;\in\;H_w\,(I)$. Then $h_{\alpha,\beta}\,\phi\;\in\;H_w\,(I)$. By Lemma 1.1(i), we have

$$h_{\alpha,\beta}\left({}_{\alpha,\beta}\tau_x\right)(y) = j_{\alpha-\beta}\,(xy)\left(h_{\alpha,\beta}\;\phi\right)(y).$$

Now we show that

$$j_{\alpha-\beta}\,(xy)\;\in\;\Lambda_m.$$

We have

$$(y^{-1}D)^m\left(j_{\alpha-\beta}\,(xy)\right) = (y^{-1}D)^m\left((xy)^{-(\alpha-\beta)}\,J_{\alpha-\beta}\,(xy)\right)$$

$$= (-1)^m\,x^{2m}\,(xy)^{-(\alpha-\beta)-m}\,J_{\alpha-\beta+m}(xy).$$

$\left|(y^{-1}D)^m\,j_{\alpha-\beta}\,(xy)\right|\;\leq\;Q_{\alpha,\beta}\,x_0^{2m}$, so that there exists $\lambda>0$ such that

$\left|e^{-\lambda w(y)}(y^{-1}D)^m j_{\alpha-\beta}\,(xy)\right|\;<\;\infty$, for every $x\;\in I$.

Here $j_{\alpha-\beta}\,(xy)\;\in\;\Lambda_m$ for fixed $x\;\in I$. But $\left(h_{\alpha,\beta}\,\phi\right)(y)\;\in\;H_w$, then $j_{\alpha-\beta}\,(xy)\,h_{\alpha,\beta}\,\phi\,(y)\;\in\;H_w$. Since $h_{\alpha,\beta}$ is an automorphism of H_w, we have ${}_{\alpha,\beta}\tau_x\,\phi\;\in\;H_w$ and the mapping $\phi\;\mapsto\;{}_{\alpha,\beta}\tau_x\,\phi$ is continuous from H_w into itself.

Theorem 3.3: If $, g\;\in\;H_w\,(I)$, then $f\,\#_{\alpha,\beta}\,g\;\in\;H_w\,(I)$.

Proof: By definition (2.4), we have

$$\eta_{k,n}\left(h_{\alpha,\beta}\left(\phi\#_{\alpha,\beta}\,\psi\right)\right)(x) = \sup_{x\in I} e^{kw(x)}\left|(x^{-1}D)^n\,h_{\alpha,\beta}\left(\phi\,\#_{\alpha,\beta}\,\psi\right)\right|.$$

Using Lemma 1.1 (ii), we have

$$\eta_{k,n}\left(h_{\alpha,\beta}\left(\phi \#_{\alpha,\beta}\psi\right)\right)(x) = \underset{x\in I}{Sup}\, e^{kw(x)}\left|(x^{-1}D)^n\left(h_{\alpha,\beta}\,\phi\right)(x)\left(h_{\alpha,\beta}\,\psi\right)(x)\right|$$

$$= \underset{x\in I}{Sup}\, e^{kw(x)}\sum_{\nu=0}^{n}\binom{n}{\nu}\left|(x^{-1}D)^{n-\nu}\left(h_{\alpha,\beta}\,\phi\right)(x)\,(x^{-1}D)^{\nu}\left(h_{\alpha,\beta}\,\psi\right)(x)\right|$$

$$= \sum_{\nu=0}^{n}\binom{n}{\nu}\underset{x\in I}{Sup}\, e^{kw(x)}\left|(x^{-1}D)^{n-\nu}\left(h_{\alpha,\beta}\,\phi\right)(x)\right|\left|\underset{x\in I}{Sup}\,(x^{-1}D)^{\nu}\left(h_{\alpha,\beta}\,\psi\right)(x)\right|$$

$$= \sum_{\nu=0}^{n}\binom{n}{\nu}\,\eta_{k,n-\nu}\left(h_{\alpha,\beta}\,\phi\right)(x)\,\eta_{0,\nu}\left(h_{\alpha,\beta}\,\psi\,(x)\right) < \infty.$$

Hence $h_{\alpha,\beta}\left(\phi \#_{\alpha,\beta}\psi\right) \in H_w(I)$. Since $h_{\alpha,\beta}$ is an automorphism of $H_w(I)$, we have $\phi \#_{\alpha,\beta}\psi \in H_w(I)$.

Thus proof is completed.

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