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# ON THE GIBBS PHENOMENON FOR NORLUND METHOD OF SUMMABILITY

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**ABSTRACT**

In this paper, we consider a monotonic non-increasing sequence $\{p_n\}$ and find the condition under which the Norlund summability method $(N, p_n)$ shows Gibbs phenomenon.

**1. INTRODUCTION:** In the theory of approximation, it is important to study about the limit of convergence of approximating function and the limit of approximant. The relating study for a discontinuous function $\phi(x)$, defined as $\phi(x) = (\pi-x)/2$, $0 < x < 2\pi$; $= 0$, $x = 0, 2\pi$, has been firstly investigated by J. W. Gibbs by taking partial sum $s_n(x)$ of the Fourier series of $\phi(x)$ in the neighbourhood of a point of discontinuity of $\phi(x)$. Since

$$\sum_{k=1}^{\infty} (\sin kx)/k = (\pi-x)/2 = \phi(x), \qquad 0 < x < 2\pi,$$

we see that the series is not uniformly convergent in the neighbourhood of $x = 0$. Let $x > 0$, we have

$$s_n(x) = (-x/2) + \int_0^x D_n(t)\,dt,$$

where $D_n(t) = \sin((n+1)/2)t/2\sin(t/2)$. Since the integral

$$(2/\pi) \int_0^{\xi} (\sin nt/t)\,dt, \qquad 0 \le \xi \le \pi,$$

is uniformly bounded in n and $\xi$, we have

$$s_n(x) + (x/2) = \int_0^{nx} (\sin t/t)\,dt + o(1), \qquad (1.1)$$

uniformly in $0 \le x \le \pi$. Thus $s_n(x)$ are uniformly bounded, but the curve of approximation overshoot the mark in the neighbourhood of $x = 0$ in the interval $(0, \pi]$ ( cf. Knopp [4], p.379 for $n = 9$). The smoothening of convergence of Fourier series is quite important for filter design (cf. Hamming [2]). More precisely, we consider the integral of $(\sin t/t)$ over the intervals $(k\pi, (k+1)\pi)$, $k = 0, 1, 2, \ldots$. We know that these integrals decrease in absolute value and are of alternating sign (cf. Zygmund [5], p. 61) for $k = 0, 1, 2, \ldots$, the curve

$$y = \int_0^x (\sin t/t)\,dt = G(x), \text{ say},$$

takes maxima with $M_1 > M_3 > M_5 > \ldots$, at the points $\pi, 3\pi, 5\pi, \ldots$, and minima $m_2, < m_4 < m_6 < \ldots$, at the points $2\pi, 4\pi, 6\pi, \ldots$. From (1.1), we obtain

$$s_n(\pi/n) \to \int_0^{\pi} (\sin t/t)\,dt > (\pi/2).$$

Thus, though $s_n(x)$ tends to $\phi(x)$ at every fixed x, $0 < x < 2\pi$, the curve $y = s_n(x)$, which pass through the point (0, 0) condense to the interval $0 \le y \le G(\pi)$ of the y-axis, the ratio of whose length to that of the interval $0 \le y \le \phi(+0) = (\pi/2)$ is

$$(2/\pi) \int_0^{\pi} (\sin t/t)\,dt = 1.179\ldots .$$

Similarly, to the left of $x = 0$, the curve $y = s_n(x)$ condense to the interval $-G(\pi) \le y \le 0$. This behaviour is called Gibbs phenomenon i.e., if the ratio $[s_n(+0)-s_n(0)]/[\phi(+0)-\phi(0)] > 1$, then $s_n(x)$ show Gibbs phenomenon in the right of $x = 0$. The generalized form of Gibbs phenomenon is described in Zygmund ([5], p. 61). The Gibbs phenomenon for $(C, \alpha)$ method, $0 < \alpha < 1$, was studied by Zygmund ([5], p.110) and the following was obtained:

**Theorem A.** There is an absolute constant $\alpha_0$, $0 < \alpha_0 < 1$, with the following property: if f(x) has a simple discontinuity at a point $\xi$, the means $\sigma_n^{\alpha}(x; f)$ show Gibbs phenomenon at $\xi$ for $\alpha < \alpha_0$ but not for $\alpha \ge \alpha_0$.

In this paper, we consider a more general method $(N, p_n)$ than $(C, \alpha)$ method, $\alpha > -1$. The concerned $(N, p_n)$ methods are those which sum the Fourier series at a point of discontinuity of the function. The following is due to Hille and Tamarkin [3]:

**Theorem B**. Let $\{p_n\}$ be a non-negative, non-increasing sequence and let $t_n(x)$ denote the $(N, p_n)$ mean of $s_n(x)$. Then for $[f(x+t)+f(x-t)-\{f(x+0)+f(x-0)\}] = o(1)$, as $t \to 0$, then $t_n(x) \to (1/2)\,[f(x+0)+f(x-0)]$ if and only if

$$\sum_{k=1}^{n} (P_k/k) \le MP_n, \; n = 1, 2, \ldots, \qquad (1.2)$$

where M is some positive constant. We know that the condition (1.2) for the sequence $\{p_n\}$ is equivalent to the condition (cf. Dikshit and Kumar [1]),

$$K \ge P_m \sum_{n=m}^{\infty} (1/nP_n), \qquad (1.3)$$

where K is some positive constant. From Lemma 1, we find that the condition (1.3) is equivalent to $(P_k/P_n) \le (k/n)^{\alpha}$, $1 \le k \le n$, for some $\alpha$ in (0, 1). Thus, a condition of the above type is natural one for considering Gibbs phenomenon of $(N, p_n)$ method. In fact, we prove the following theorem:

**Theorem 1.** Let $\{p_n\}$ be a non-negative and non-increasing sequence. Let $\alpha$ be a number such that $(P_k/P_n) \le (k/n)^{\alpha}$, $1 \le k \le n$, then there exists a constant $\alpha_0$, $0 < \alpha_0 < 1$ such that the $(N, p_n)$ method shows Gibbs phenomenon for $\alpha < \alpha_0$, but not for $\alpha \ge \alpha_0$ at a point of simple discontinuity $\xi$ of f(x).

We need the following lemmas:

Lemma 1. Let $\{p_n\}$ be a non-negative and non-increasing sequence and let

$$P_m \sum_{n=m}^{\infty} (1/nP_n) \le K, \qquad m = 1, 2, \ldots,$$

where K is some positive constant, then $(P_m/P_n) \le (m/n)^{\delta}$, for some $0 < \delta \le 1$, $1 \le m \le [n/c]$, c is some fixed positive integer.

Proof. For any integer k, we have

$$K \ge P_m \sum_{n=m}^{\infty} (1/nP_n) \ge P_m \sum_{n=m}^{km} (1/nP_n)$$

$$\ge (P_m/P_{km}) \log k.$$

That is

$$(P_{km}/P_m) \ge (\log_4 k \log 4^e/K) \ge 4, \text{ for large } k \ge k_0, \qquad (1.4)$$

We take for convenience $k_0 \ge 4$. For a given sufficiently large n, we can find a fixed integer $c \ge k_0$, and r such that

$$c^{r+(1/2)} m \le n < c^{r+1} m.$$

We have

$$(P_n/P_m) = (P_n/P_c r_m)(P_c r_m/P_m) \ge (P_n/P_c r_m)\, 4^r, \qquad (1.5)$$

by a repeated application of the fact that $P_{km}/P_m \ge 4$.

We can find a number $\mu$, $(1/2) \le \mu < 1$, such that $n = c^{r+\mu} m$. We have

$$r = \log_4 (n/m)^{\delta} - \mu, \qquad (1.6)$$

where $\delta = (1/\log_4 c)$. Obviously, $\delta \le 1$.

From (1.5) and (1.6), we get

$$(P_n/P_m) \ge \frac{P_{c^{r+\mu} m}}{P_{c^r m}} (4)^{\log_4 (n/m)^{\delta} - \mu}$$

$$= \frac{P_{c^{r+\mu} m}}{P_{c^r m}} (4)^{-\mu} (n/m)^{\delta}. \qquad (1.7)$$

Again from (1.3), we have

$$K \ge P_{c^r m} \sum_{n=c^r m}^{c^{r+\mu} m} (1/nPn)$$

$$P_{c^r m}$$

$$\geq \frac{}{P_c^{r+\mu}m} \log c^{\mu},$$

that is

$$\frac{P_c^{r+\mu}m}{P_c^{r}m} \geq (\log c^{\mu}/K). \tag{1.8}$$

Now, from (1.7) and (1.8), we obtain

$$(P_n/P_m) \geq \frac{\log c^{\mu}}{K} 4^{-\mu} (n/m)^{\delta}$$

$$\geq 4\mu 4^{-\mu} (n/m)^{\delta} \geq (n/m)^{\delta}, \tag{1.9}$$

by the fact that $4^{\mu} \leq 4\mu$, for $(1/2) \leq \mu < 1$. Thus (1.9) shows that $(P_m/P_n) \leq (m/n)^{\delta}$, $0 < \delta \leq 1$, $1 \leq m \leq [n/c]$.

This proves the lemma.

Lemma 2. Given any $m > 0$, there exists a $\delta(m) > 0$ and $n_0(m)$ such that

$$\sigma_n(x) < (\pi/2)-\delta \quad \text{for } 0 \leq x \leq (m/n),\ n > n_0.$$

Lemma 2 is contained in Zygmund ([5], p.111).

**Proof of the Theorem**. Since the partial sum $s_n(x)$ is uniformly summable $(N, p_n)$ at every point of continuity (cf. Hille and Tamarkin [3]), so to prove the theorem, we prove it for the function $f(x) \sim \sin x + (\sin 2x/2) + \ldots$, at $\xi = 0$. Observing that $s_n' = \cos x + \cos 2x + \ldots$, we get

$$s_n(x) = \int_0^x \left(\sum_{k=1}^{n} \cos kt\right) dt = ((\pi-x)/2) - \int_x^{\pi} D_n(t)dt,$$

and

$$t_n^p(x) = ((\pi-x)/2) - (1/2P_n) \sum_{k=0}^{n} \left(\int_x^{\pi} p_{n-k} \frac{\sin(k+(1/2))t}{\sin(t/2)} dt\right).$$

We write

$$(1/2P_n)\left(\sum_{k=0}^{[n/2]} + \sum_{k=[n/2]+1}^{n}\right) p_{n-k}(\sin(k+(1/2)t)/\sin(t/2))$$

$$= \Sigma_1 + \Sigma_2, \text{ say.} \tag{1.10}$$

Applying Abel's Lemma, we find that

$$|\Sigma_1| \leq 1/n(\sin(t/2))^2,$$

Hence

$$\left|\int_x^{\pi} \Sigma_1\, dt\right| \leq (2/n)\cot(x/2). \tag{1.11}$$

Again using mean value theorem, we have for some $x < \xi < \pi$

$\pi$

$$\left| \int_0 \Sigma_2\, dt \right| \le K\left( P_{1/x}/nP_n \sin(x/2)\right), \tag{1.12}$$

since $P_{1/\xi} \le P_{1/x}$ for $x < \xi$.

Combining (1.10), (1.11) and (1.12), we get

$$t_n^p(x) \le (\pi - x)/2 + (2/n)\cot(x/2) + K\left(P_{1/x}/nP_n \sin(x/2)\right). \tag{1.13}$$

By the hypothesis that $(P_k/P_n) \le (k/n)^\alpha$, $0 < \alpha < 1$, we see that the second term in (1.13) dominate the last term. Thus, if nx is sufficiently large, say $nx > m$, $n \ge n_1$ and $nx^2 > 1$, we find that

$$\left| t_n^p(x) \right| \le (\pi/2) \quad \text{for } (n/m) \le x \le \pi. \tag{1.14}$$

Now, we show that if the sequence $\{p_n\}$ is suitably chosen then the inequality (1.14) is true for other values of x, i.e., for $0 \le x \le (m/n)$. To see this, we consider $t_n^p(x) - \sigma_n(x)$, where $\sigma_n(x)$ denote the (C, 1) mean of $s_n(x)$, we have

$$\left| t_n^p(x) - \sigma_n(x) \right| = \left| \sum_{k=0}^{n} \frac{P_{n-k}}{P_n} \frac{\sin kx}{k} - \sum_{k=0}^{n} \frac{n-k+1}{n+1} \frac{\sin kx}{k} \right|$$

$$\le x \sum_{k=0}^{n} \frac{n-k+1}{P_n} \left( \frac{P_{n-k}}{n-k+1} - \frac{P_n}{n+1} \right),$$

since $(P_n/n)$ is non-increasing for $\{p_n\}$. We have

$$\left| t_n^p(x) - \sigma_n(x) \right| \le x\left[ (P_n^1/P_n) - ((n+2)/2)\right].$$

Now,

$$(P_n^1/P_n) = \sum_{k=0}^{n} (P_k/P_n) = \sum_{k=0}^{n} \int_k^{k+1} (P_x/P_n)\, dx$$

$$\le \int_0^{n+1} (x/n)^\alpha\, dx = \left[ (n+1)^{\alpha+1}/(\alpha+1)n^\alpha \right].$$

We have

$$\left| t_n^p(x) - \sigma_n(x) \right| \le x\left[ \frac{(n+1)^{\alpha+1}}{(\alpha+1)n^\alpha} - \frac{n+2}{2} \right]$$

$$= \frac{nx(1-\alpha)}{2(\alpha+1)} + x\left[ \frac{(n+1)^{\alpha+1} - n^{\alpha+1}}{(\alpha+1)n^\alpha} - 1\right].$$

Since $(n+1)^{\alpha+1} - n^{\alpha+1} \le (2n)^\alpha$ and $2^\alpha \le \alpha+1$ for $0 \le \alpha \le 1$, we have

$$\left| t_n^p(x) - \sigma_n(x) \right| \le \left[ nx(1-\alpha)/2(\alpha+1) \right].$$

That is

$$t_n^p(x) \le \sigma_n(x) + (nx/2)(1-\alpha).$$

By Lemma 2, we have

$$t_n^p(x) \leq (\pi/2) - \delta(m) + (m_1/2)(1-\alpha), \quad 0 \leq nx \leq m_1.$$

Now, if we take $\alpha$, such that $(1-\alpha)m_1/2 - \delta(m_1) < 0$, then

$$t_n^p(x) \leq (\pi/2), \text{ for } 0 \leq nx \leq m_1.$$

In order to show that for positive and small enough $\alpha$, the Gibbs phenomenon occurs, and it does not occur for $\alpha \geq 1$, we consider the difference $t_n^p(x) - s_n(x)$. We have

$$\left| t_n^p(x) - s_n(x) \right| \leq x\,(n - (P_n^1/P_n)) < nx\alpha.$$

Thus

$$\left| t_n^p(\pi/n) - s_n(\pi/n) \right| \leq \pi\alpha, \qquad \text{for } 0 < \alpha < 1.$$

Consequently,

$$s_n(\pi/n) - \pi\alpha \leq t_n^p(\pi/n) \leq \pi\alpha + s_n(\pi/n).$$

From the above inequality, we see that for small $\alpha$

$$\liminf_{n\to\infty} t_n^p(\pi/n) > (\pi/2),$$

by the fact that $s_n(\pi/n)$ tends to a limit greater than $(\pi/2)$.

Hence the Gibbs phenomenon occurs for small values of $\alpha$. This proves that there exists $\alpha_0$, $0 < \alpha_0 < 1$, such that for $\alpha < \alpha_0$ the Gibbs phenomenon exists, while for $\alpha > \alpha_0$ it does not exist.

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